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Research Article

New Operators in Ideal Topological Spaces and Their Closure Spaces

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Abstract

In this paper, we introduce two operators associated with Ψ^* and $*^\Psi$ operators in ideal topological spaces and discuss the properties of these operators. We give further characterizations of Hayashi-Samuel spaces with the help of these two operators. We also give a brief discussion on homeomorphism of generalized closure spaces which were induced by these two operators.

Keywords

Ideal topological spaces, Δ-operator, ∇-operator, Hayashi-Samuel space, isotonic spaces, homeomorphism.

1. INTRODUCTION

The study of local function on ideal topological space was introduced by Kuratowski [1] and Vaidyanathswamy [2]. The mathematicians like Jankovic and Hamlett [3, 4], Samuel [5], Hayashi [6], Hashimoto [7], Newcomb [8], Modak [9, 10], Bandyopadhyay and Modak [11, 12], Noiri and Modak [13], Al-Omari et al. [14, 15, 16, 17] have enriched this study. Natkaniec in [18] have introduced the complement of local function and it is called Ψ-Operator. In an ideal topological space $(X, \tau, \mathcal{I})$, the local function $()^*$ is defined as: $A^*(\mathcal{I}, \tau)$ (or, simply, A^*) = $\{x \in X : U_x \cap A \notin \mathcal{I}\}$, where $U_x \in \tau(x)$, the collection of all open sets containing x. Its

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complement function, that is, Ψ -operator is defined as: $\Psi(A) = X \setminus (X \setminus A)^*$. Using these two set functions, $()^*$ and Ψ, Modak and Islam [19, 20] have introduced two moreoperators in the ideal topological spaces and they are: $*^\Psi(A) = \Psi(A^*) = X \setminus (X \setminus A^*)^*$ and $\Psi^*(A) = (\Psi(A))^* = \{x \in X : U_x \cap \Psi(A) \notin \mathcal{I}\}$, where $U_x \in \tau(x)$.

Following example shows that the values of the operators Ψ^* and $*^\Psi$ are not the same:

Example 1.1. Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{c\},X\}$ and $\mathcal{I} = \{\emptyset,\{c\}\}$. Then, $*^\Psi(X) = \Psi(X^*)$ $= \Psi(\{a,b\}) = X \setminus (\{c\})^* = X$ and $(\Psi(X))^* = X^* = \{a,b\}$. Therefore, $\Psi^*(X) \neq *^\Psi(X)$.

The value of the operator $*^\Psi$ is an open set and the value of the operator Ψ^* is a closed set. In this paper, we further consider the operators using joint operators Ψ^* and $*^\Psi$ simultaneously and shall define two more operators using of Ψ^* and $*^\Psi$ which is Δ and meet of Ψ^* and $*^\Psi$ which is ∇. We also consider the values of these two operators on various ideal topological spaces as well as various subsets of the ideal topological space. We also give a bunch of characterization of Hayashi-Samuel space. An ideal topological space $(X,\tau,\mathcal{I})$ is called Hayashi-Samuel space [21], if $\tau \cap \mathcal{I} = \{\emptyset\}$. Theauthors Hamlett and Jankovi$\acute{c}$ [3] called it by the name of τ -boundary, whereas the authors Dontchev, Ganster and Rose [22] called it by the name of codense ideal. In the study of ideal topological spaces, it played an important role. Two well known Hayashi-Samuel spaces are: Let τ be a topology on a set X, then $(X,\tau,\{\emptyset\})$ is a Hayashi-Samuel space and if $\mathcal{I}_n$ is the collection of all nowhere dense subsets of (X,τ), then $(X,\tau,\mathcal{I}_n)$ is also a Hayashi-Samuel space.

Further, we also give the topological properties of the generalized closure spaces [23, 24] induced by the above mentioned operators Δ and ∇.

Now we shall give a few words about generalized closure spaces. The study of closure spaces was introduced by Habil and Elzenati [23] in 2003 and Stadler [24] in 2005. Generalized closure space is the generalization of closure space and its definition is as follows:

Definition 1.2. Let X be a set, $\wp(X)$ be the power set of X and $cl : \wp(X) \to \wp(X)$ be any arbitrary set-valued set-function, called a closure function. We call $cl(A)$ the closure of A, and we call the pair (X,cl) a generalized closure space (see [23, 24]).

Consider the following axioms (see [23, 24]) of the closure function for all $A, B, A_\lambda \in \wp(X)$, Λ is an index set:

The closure function in a generalized closure space (X, cl) is called:

(K0) grounded, if $cl(\varnothing) = \varnothing$.

(K1) isotonic, if $A \subseteq B$ implies $cl(A) \subseteq cl(B)$.

(K2) expanding, if $A \subseteq cl(A)$.

(K3) sub-additive, if $cl(A \cup B) \subseteq cl(A) \cup cl(B)$.

(K4) idempotent, if $cl(cl(A)) = cl(A)$.

(K5) additive, if $\bigcup_{\lambda\in\Lambda} cl(A_\lambda) = cl(\bigcup_{\lambda\in\Lambda}(A_\lambda))$.

Definition 1.3. [24, 25, 26] A pair (X, cl) is said to be an isotonic space if it satisfies the axioms (K0) and (K1). If an isotonic space (X, cl) satisfies (K2), then it is called a neighbourhood space. A closure space that satisfies (K4), is called a neighbourhood space. A topological space, that satisfies (K3), is a closure space.

'*int*' is the complement function of the closure function '*cl*' and it is defined as:

$$\text{int}(A) = X \setminus cl(X \setminus A), \text{ for } A \subseteq X.$$

2. Δ Operator

Definition 2.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We define the operator $\Delta : \wp(X) \to \wp(X)$ as:

$$\Delta(A) = \Psi^*(A) \cup {*^{\Psi}}(A), \text{ for } A \subseteq X.$$

Observe that, for $A \subseteq X$, $\Delta(A)$ is the union of an open set and a closed set.

The next example shows that union of an open set and a closed set is not always an expression of $\Delta(A)$, for any $A \subseteq X$.

Example 2.2. Let $X = \{a,b,c\}$, $\tau = \{\varnothing, \{a\}, \{a,b\}, X\}$ and $\mathcal{I} = \{\varnothing, \{b\}\}$. Let $A_1 = \{a\}$ and $A_2 = \{c\}$. Then, A_1 is open and A_2 is closed. Then $A_1 \cup A_2 = \{a,c\}$. Now $(\Psi(\varnothing))^* = \varnothing$ $= (\Psi(\{b\}))^* = (\Psi(\{c\}))^* = (\Psi(\{b,c\}))^*$, $(\Psi(\{a\}))^* = X = (\Psi(\{a,b\}))^* = (\Psi(\{a,c\}))^*$

$= (\Psi(X))^*$ and $\Psi(\varnothing^*) = \varnothing = \Psi((\{b\})^*) = \Psi((\{c\})^*) = \Psi((\{b,c\})^*)$, $\Psi((\{a\})^*) = X$ $= \Psi((\{a,b\})^*) = \Psi((\{a,c\})^*) = \Psi(X^*)$. So there is no $T \in \wp(X)$ such that $\Delta(T) = A_1 \cup A_2$.

If $\mathcal{I} = \{\varnothing\}$, then $\Delta(A) = Int(Cl(A)) \cup Cl(Int(A))$ (where '*Int*' and '*Cl*' denote the interior and closure operator of (X,τ) respectively) and if $\mathcal{I} = \mathcal{I}_n$, then

$\Delta(A) = [Int(Cl(Int(Cl(Int(Cl(A))))))] \cup [Cl(Int(Cl(Int(Cl(Int(A))))))]$
$= Int(Cl(A)) \cup Cl(Int(A))$.

Therefore, the value of Δ, for any subset A of X on $(X,\tau,\{\varnothing\})$ and $(X,\tau,\mathcal{I}_n)$ are equal.

The operator Δ is not grounded and it follows from the following example:

Example 2.3. Let $X = \{a,b,c,d\}$, $\tau = \{\varnothing,\{a\},X\}$ and $\mathcal{I} = \{\varnothing,\{a\}\}$. Then, $\Delta(\varnothing) = \Psi^*(\varnothing) \cup *^{\Psi}(\varnothing) = \varnothing \cup \{a\} = \{a\} \neq \varnothing$. So, the operator Δ is not grounded.

Theorem 2.4. An ideal topological space $(X,\tau,\mathcal{I})$ is Hayashi-Samuel, if and only if, the operator $\Delta : \wp(X) \to \wp(X)$ is grounded.

Proof. Suppose that $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then, $X = X^*$ [4].
Now, $\Delta(\varnothing) = \Psi^*(\varnothing) \cup *^{\Psi}(\varnothing) = (X \setminus X^*)^* \cup (X \setminus X^*) = \varnothing^* \cup \varnothing = \varnothing$.

Conversely suppose that $\Delta(\varnothing) = \varnothing$. Then $\Psi^*(\varnothing) \cup *^{\Psi}(\varnothing) = \varnothing$, implies, $(\Psi(\varnothing))^* \cup \Psi(\varnothing^*) = \varnothing$, implies, $(X \setminus X^*)^* \cup (X \setminus X^*) = \varnothing$. Thus, $X \setminus X^* = \varnothing$ and $(X \setminus X^*)^* = \varnothing$. Hence, $(X,\tau,\mathcal{I})$ is a Hayashi-Samuel space.

We recall following definition:

Definition 2.5. Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then, A is said to be a Ψ^*-set [9] (resp. Ψ-C set [12], regular open set [27]) if $A \subseteq (\Psi(A))^*$ (resp. $A \subseteq Cl(\Psi(A)), A = Int(Cl(A))$).

The collection of all Ψ^*-sets (resp. Ψ-C sets) in $(X,\tau,\mathcal{I})$ is denoted as $\Psi^*(X,\tau)$ (resp. $\Psi(X,\tau)$).

Corollary 2.6. In an ideal topological space $(X,\tau,\mathcal{I})$, the following properties are equivalent:

1. $(X,\tau,\mathcal{I})$ is a Hayashi-Samuel space [20];
2. $\Psi(\varnothing)=\varnothing$ [20];
3. if $A\subseteq X$ is closed, then, $\Psi(A)\setminus A=\varnothing$ [20];
4. $*^{\Psi}:\wp(X)\to\wp(X)$ is grounded;
5. if $A\subseteq X$, then, $Int(Cl(A))=\Psi(Int(Cl(A)))$ [20];
6. A is regular open, $A=\Psi(A)$ [20];
7. Δ is grounded;
8. if $U\in\tau$, then, $\Psi(U)\subseteq Int(Cl(U))\subseteq U^*$ [20];
9. if $I\in\mathcal{I}$, then, $\Psi(I)=\varnothing$ [20];
10. $\Psi^*(X,\tau)=\Psi(X,\tau)$ [20];
11. $\Psi^*(A)=Cl(\Psi(A))$, for each $A\subseteq X$ [20];
12. $G\subseteq G^*$, for each $G\in\tau$;
13. $\Psi^*(X)=X$;
14. if $J\in\mathcal{I}$, then, $Int(J)=\varnothing$.

Proof. Follows from Theorem 2.4 and Corollary 2.18 of [20].

Theorem 2.7. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then the operator $\Delta:\wp(X)\to\wp(X)$ is isotonic.

Proof. Follows from the following facts:

(i) The operator * is isotonic.

(ii) The operator ψ is isotonic.

The following example shows that the operator Δ is not expanding.

Example 2.8. Let $X=\{a,b,c\}$, $\tau=\{\varnothing,X\}$ and $\mathcal{I}=\{\varnothing,\{a\}\}$. Let $A=\{a\}$. Then, $\Psi^*(A)=\varnothing=*^{\Psi}(A)$. Thus, $\Delta(A)=\Psi^*(A)\cup*^{\Psi}(A)=\varnothing$. Hence, $A\not\subseteq\Delta(A)$.

Theorem 2.9. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then for $A,B\in\wp(X)$, $\Delta(A)\cup\Delta(B)\subseteq\Delta(A\cup B)$.

Proof. Let $A,B\in\wp(X)$. Since, $A\subseteq A\cup B$ and Δ is isotonic, hence, $\Delta(A)\subseteq\Delta(A\cup B)$. Similarly, $\Delta(B)\subseteq\Delta(A\cup B)$. Hence $\Delta(A)\cup\Delta(B)\subseteq\Delta(A\cup B)$.

Since $Int(Cl(A\cup B))\neq Int(Cl(A))\cup Cl(Int(A))\cup Cl(Int(B))\cup Int(Cl(B))$, the operator Δ is not sub-additive, and hence it is not additive.

Theorem 2.10. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then $\Delta(A)\subseteq A^*$, for any $A\subseteq X$.

Proof. Follows from the following facts:

(i) $\Psi(A^*)\subseteq A^*$, for any $A\in\wp(X)$.

(ii) $(\Psi(A))^*\subseteq A^*$, for any $A\in\wp(X)$.

Corollary 2.11. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then $\Delta(A)\subseteq Cl^*(A)$, for any $A\subseteq X$.

Corollary 2.12. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then $\Delta(X)=X$.

Following example shows that the converse of the Corollary 2.12 does not hold, in general.

Example 2.13. Let $X=\{a,b,c\}$, $\tau=\{\varnothing,\{c\},X\}$ and $\mathcal{I}=\{\varnothing,\{c\}\}$. Then, $*^{\Psi}(X)=\Psi(X^*)=\Psi(\{a,b\})=X\setminus(\{c\})^*=X$ and $(\Psi(X))^*=X^*=\{a,b\}$. Therefore, $\Delta(X)=\Psi^*(X)\cup*^{\Psi}(X)=X$ but $(X,\tau,\mathcal{I})$ is not a Hayashi-Samuel space.

Theorem 2.14. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then for $U\in\tau$, $Int(Cl(U))\subseteq\Delta(U)\subseteq U^*=Cl(U)$.

Proof. We have, $\Delta(U)=\Psi^*(U)\cup*^{\Psi}(U)=(\Psi(U))^*\cup\Psi(U^*)=Cl(\Psi(U))\cup\Psi(Cl(U))$ [13] $=Cl(\Psi(U))\cup[X\setminus(X\setminus Cl(U))^*]=[X\setminus Cl\Psi(U)\cup Cl(X\setminus Cl(U))]$. This implies that $Int(Cl(U))\subseteq\Delta(U)$.

Further, from Theorem 2.10, $\Delta(U) \subseteq U^* \subseteq Cl(U)$. Thus $Int(Cl(U)) \subseteq \Delta(U) \subseteq U^* = Cl(U)$.

The authors Janković and Hamlett have introduced a new topology $\tau^*(\mathcal{I})$ [4] from $(X, \tau, \mathcal{I})$. Its closure operator is denoted as Cl^* [4].

Theorem 2.15. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $J \in \mathcal{I}$. Then, $\Delta(J) = Cl^*(X \setminus X^*)$.

Proof. Let $J \in \mathcal{I}$. Then, $(X \setminus J)^* = X^*$ [4]. $\Delta(J) = \Psi^*(J) \cup *^{\Psi}(J) = (\Psi(J))^* \cup \Psi(J^*)$ $= (X \setminus (X \setminus J)^*)^* \cup \Psi(\varnothing) = (X \setminus X^*)^* \cup (X \setminus X^*) = Cl^*(X \setminus X^*)$ [12].

Corollary 2.16. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space and $J \in \mathcal{I}$. Then, $\Delta(J) = \varnothing$.
It is not necessary that $\Delta(A) = \varnothing$ implies $A \in \mathcal{I}$.

Example 2.17. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\varnothing, \{b\}\}$. Let $A = \{b, c\} \notin \mathcal{I}$. Then, $\Psi^*(A) = *^{\Psi}(A) = \varnothing$. So, $\Delta(A) = \varnothing$. This example shows that $\Delta(A) = \varnothing$ but $A \notin \mathcal{I}$.

Corollary 2.18. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then,
$\Delta(A \cup J) = \Delta(A \setminus J) = \Delta(A)$, for $A \subseteq X$, $J \in \mathcal{I}$.

Proof. Obvious from [3] and [4].

3. ∇ Operator

In this section, we shall define another operator ∇ and discuss the role of ∇ in Hayashi-Samuel spaces.

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We define the operator $\nabla : \wp(X) \to \wp(X)$ as:

$$\nabla(A) = \Psi^*(A) \cap *^{\Psi}(A), \text{ for } A \subseteq X.$$

It is obvious that for a subset A of X, the value $\nabla(A)$ is the intersection of a closed set and an open set, since, $\Psi^*(A)$ is a closed set and $*^{\Psi}(A)$ is an open set. Thus, $\nabla(A)$ is a locally closed set in (X,τ) for any $A\in\wp(X)$.

Example 3.2. Let $X=\{a,b,c\}, \tau=\{\varnothing,\{a\},\{a,b\},X\}$ and $\mathcal{I}=\{\varnothing,\{b\}\}$. Also let $H=\{b\}$. Then $H=\{a,b\}\cap\{b,c\}$. So H is a locally closed set. Now, $(\Psi(\varnothing))^*=\varnothing=(\Psi(\{b\}))^*$ $=(\Psi(\{c\}))^*=(\Psi(\{b,c\}))^*, (\Psi(\{a\}))^*=X=(\Psi(\{a,b\}))^*=(\Psi(\{a,c\}))^*=(\Psi(X))^*$ and
$\Psi(\varnothing^*)=\varnothing=\Psi((\{b\})^*)=\Psi((\{c\})^*)=\Psi((\{b,c\})^*), \Psi((\{a\})^*)=X=\Psi((\{a,b\})^*)$
$=\Psi((\{a,c\})^*)=\Psi(X^*)$.
So, there does not exist any set $A,B\subseteq X$, such that H can be expressed as $H=(\Psi(A))^*\cap\Psi(B^*)$. Therefore, we conclude that locally closed set cannot be decomposed by the operators Ψ^* and $*^{\Psi}$.

If $\mathcal{I}=\{\varnothing\}$, then $\nabla(A)=\Psi^*(A)\cap *^{\Psi}(A)=(\Psi(A))^*\cap\Psi(A^*)=[Int(Cl(A))]\cap[Cl(Int(A))]$.

If $\mathcal{I}=\mathcal{I}_n$, then $\nabla(A)=\Psi^*(A)\cap *^{\Psi}(A)=(\Psi(A))^*\cap\Psi(A^*)=[Int(Cl(A))]\cap[Cl(Int(A))]$
$=[Int(Cl(Int(Cl(Int(Cl(A))))))]\cap[Cl(Int(Cl(Int(Cl(Int(A))))))]$.
Moreover, $X\setminus\Delta(A)=\nabla(X\setminus A)$.

The value of ∇ on a subset A of X on the spaces $(X,\tau,\{\varnothing\})$ and $(X,\tau,\mathcal{I}_n)$ are equal.

Theorem 3.3. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then the operator $\nabla:\wp(X)\to\wp(X)$ is grounded.

Proof. Obvious from the facts that:
(i) $X=X^*$, for the Hayashi-Samuel space $(X,\tau,\mathcal{I})$.
(ii) $*^{\Psi}(\varnothing)=\varnothing$.
(iii) $\Psi^*(\varnothing)=\varnothing$.

The following example shows that the converse of the above theorem is not true, in general:

Example 3.4. Let $X=\{a,b\}$, $\tau=\{\varnothing,\{a\},X\}$ and $\mathcal{I}=\{\varnothing,\{a\}\}$.

Then $\nabla(\varnothing)=\Psi^*(\varnothing)\cap *^{\Psi}(\varnothing)=\varnothing\cap\{a\}=\varnothing$, but $(X,\tau,\mathcal{I})$ is not a Hayashi-Samuel space.

Theorem 3.5. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then the operator $\nabla:\wp(X)\to\wp(X)$ is isotonic.

Proof. Since, both the operators * and Ψ are isotonic, then ∇ is isotonic.

The following Example shows that the operator ∇ is not expanding.

Example 3.6. Let $X=\{a,b,c\}$, $\tau=\{\varnothing,X\}$ and $\mathcal{I}=\{\varnothing,\{a\}\}$. Let $A=\{a\}$.

Then, $\Psi^*(A)=\varnothing=*^{\Psi}(A)$. Thus, $\nabla(A)=\Psi^*(A)\cap *^{\Psi}(A)=\varnothing$. Hence, $A\nsubseteq\nabla(A)$.

The following example shows that the operator ∇ is not subadditive.

Example 3.7. Let $X=\{a,b,c,d\}$, $\tau=\{\varnothing,\{a\},\{b\},\{a,b\},X\}$ and $\mathcal{I}=\{\varnothing,\{c\}\}$. Let $A=\{a\}$ and $B=\{b\}$. Then, $\Psi^*(A)=\{a,c,d\}$, $*^{\Psi}(A)=\{a\}$ and $\Psi^*(B)=\{b,c,d\}$, $*^{\Psi}(B)=\{b\}$. So $\nabla(A)=\Psi^*(A)\cap *^{\Psi}(A)=\{a\}$ and $\nabla(B)=\Psi^*(B)\cap *^{\Psi}(B)=\{b\}$. So, $\nabla(A)\cup\nabla(B)=\{a,b\}$. Also, $\Psi^*(A\cup B)=X$ and $*^{\Psi}(A\cup B)=X$. Thus, $\nabla(A\cup B)=\Psi^*(A\cup B)\cap *^{\Psi}(A\cup B)=X$. Therefore, $\nabla(A\cup B)\nsubseteq\nabla(A)\cup\nabla(B)$. Hence, ∇ is not subadditive.

Remark 3.8. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then the operator $\nabla:\wp(X)\to\wp(X)$ is not additive.

However following holds:

Theorem 3.9. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then for $A,B\in\wp(X)$, $\nabla(A)\cup\nabla(B)\subseteq\nabla(A\cup B)$.

Proof. Let $A, B \in \wp(X)$. Since, $A \subseteq A \cup B$ and ∇ is isotonic, then, $\nabla(A) \subseteq \nabla(A \cup B)$. Similarly, $\nabla(B) \subseteq \nabla(A \cup B)$. Hence, $\nabla(A) \cup \nabla(B) \subseteq \nabla(A \cup B)$.

Theorem 3.10. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then $\nabla(A) \subseteq A^*$, for any $A \subseteq X$.

Proof. It is obvious from the following facts:

(i) $*^{\Psi}(A) \subseteq A^*$, for the Hayashi-Samuel space $(X, \tau, \mathcal{I})$.

(ii) $\Psi^*(A) \subseteq A^*$, for the Hayashi-Samuel space $(X, \tau, \mathcal{I})$.

Corollary 3.11. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then $\nabla(A) \subseteq Cl^*(A)$, for any $A \subseteq X$.

Corollary 3.12. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then

1. $\Delta(A) \cup \nabla(A) \subseteq A^*$, for any $A \subseteq X$.
2. $\Delta(A) \cap \nabla(A) \subseteq A^*$, for any $A \subseteq X$.

Theorem 3.13. An ideal topological space $(X, \tau, \mathcal{I})$ is Hayashi-Samuel, if and only if, $\nabla(X) = X$.

Proof. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then $X^* = X$.

Then, $\nabla(X) = \Psi^*(X) \cap *^{\Psi}(X) = [(X \setminus (X \setminus X)^*)^*] \cap [X \setminus (X \setminus X^*)^*] = X^* \cap X = X$.

Conversely suppose that $X = \nabla(X) = \Psi^*(X) \cap *^{\Psi}(X) = (\Psi(X))^* \cap \Psi(X^*)$

$= [X \setminus (X \setminus X)^*]^* \cap [X \setminus (X \setminus X^*)^*] = [X \setminus (X \setminus X^*)^*] \cap X^* \subseteq X^*$. Thus, $X = X^*$, and hence the space is Hayashi-Samuel.

Corollary 3.14. In an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:

1. $(X, \tau, \mathcal{I})$ is a Hayashi-Samuel space [20];
2. $\Psi(\varnothing) = \varnothing$ [20];
3. if $A \subseteq X$ is closed, then, $\Psi(A) \setminus A = \varnothing$ [20];
4. $*^{\Psi} : \wp(X) \to \wp(X)$ is grounded;

5. if $A \subseteq X$, then, $Int(Cl(A)) = \Psi(Int(Cl(A)))$ [20];
6. A is regular open, $A = \Psi(A)$ [20];
7. Δ is grounded;
8. $\nabla(X) = X$;
9. if $U \in \tau$, then, $\Psi(U) \subseteq Int(Cl(U)) \subseteq U^*$ [20];
10. if $I \in \mathcal{I}$, then, $\Psi(I) = \varnothing$ [20];
11. $\Psi^*(X,\tau) = \Psi(X,\tau)$ [20];
12. $\Psi^*(A) = Cl(\Psi(A))$, for each $A \subseteq X$ [20];
13. $G \subseteq G^*$, for each $G \in \tau$;
14. $\Psi^*(X) = X$;
15. if $J \in \mathcal{I}$, then, $Int(J) = \varnothing$.

Corollary 3.15. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space such that $\nabla(X) = X$. Then, $\Psi^*(X) = X$ and $*^{\Psi}(X) = X$.

Proof. Follows from the fact that, $X = \nabla(X) \subseteq \Psi^*(X) \subseteq X^*$ and $X = \nabla(X) \subseteq *^{\Psi}(X) \subseteq X^*$.

Theorem 3.16. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$,
$Int(Cl(U)) \subseteq \nabla(U)$.

Proof. We have

$$\nabla(U) = \Psi^*(U) \cap *^{\Psi}(U) = (\Psi(U))^* \cap \Psi(U^*) = Cl(\Psi(U)) \cap \Psi(Cl(U))$$
$$= Cl[X \setminus (X \setminus U)^*] \cap [X \setminus (X \setminus Cl(U))^*]$$
$$\supseteq [X \setminus Int(Cl(X \setminus U))] \cap [X \setminus Cl(X \setminus Cl(U))]$$
$$= [X \setminus (X \setminus Cl(U))] \cap Int(Cl(U)) = Cl(U) \cap Int(Cl(U)) = Int(Cl(U)) .$$

Corollary 3.17. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then for $U \in \tau$,
$Int(Cl(U)) \subseteq \nabla(U) \subseteq U^* = Cl(U)$.

Theorem 3.18. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space and $J \in \mathcal{I}$. Then, $\nabla(J) = \varnothing$.

Proof. Let $J \in \mathcal{I}$. Then, $J^* = \varnothing$ [4]. Now, $\nabla(J) = \Psi^*(J) \cap *^{\Psi}(J) = (\Psi(J))^* \cap \Psi(J^*)$ $= (X \setminus (X \setminus J)^*)^* \cap \Psi(\varnothing) = (X \setminus X)^* \cap (X \setminus X^*) = \varnothing$.

The converse of this theorem is not true in general.

Example 3.19. Let $X = \{a,b\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Let $J = \{a\}$. Then, $\nabla(J) = \Psi^*(J) \cap *^{\Psi}(J) = \varnothing \cap \{a\} = \varnothing$. Here the space $(X, \tau, \mathcal{I})$ is not a Hayashi-Samuel space.

Corollary 3.20. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space and $J \in \mathcal{I}$, then $\nabla(J) = \Delta(J) = \varnothing$.

Corollary 3.21. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, for $A \subseteq X, J \in \mathcal{I}$, $\nabla(A \setminus J) = \nabla(A \cup J) = \nabla(A)$.

Proof. Obvious from [3] and [4].

Lemma 3.22. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then, for $A \subseteq X$

1. $*^{\Psi}(A) = X \setminus \Psi^*(X \setminus A)$.
2. $*^{\Psi}(X \setminus A) = X \setminus \Psi^*(A)$.

More general relation between Δ and ∇ is:

Theorem 3.23. Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then for $A \subseteq X$, $\nabla(A) = X \setminus \Delta(X \setminus A)$.

Proof. We have

$$X \setminus \nabla(A) = X \setminus [\Psi^*(A) \cap *^{\Psi}(A)] = [X \setminus \Psi^*(A)] \cup [X \setminus *^{\Psi}(A)]$$
$$= *^{\Psi}(X \setminus A) \cup [X \setminus (X \setminus \Psi^*(X \setminus A))] = [\Psi^*(X \setminus A) \cup *^{\Psi}(X \setminus A)] = \Delta(X \setminus A).$$

4. Spaces induced by Δ and ∇

In generalized closure space (X, cl), two concepts were defined: one is closure preserving [26] and other is continuity [26]. But fortunately, two concepts are coincident in the isotonic space [24, 26]. Here we define continuity in isotonic space.

Definition 4.1. [24, 26] Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces. A function $f : X \to Y$ is continuous if $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$, for all $B \in \wp(Y)$.

In isotonic spaces, (X, cl_X) and (Y, cl_Y), we can represent the continuity by the following way:

Definition 4.2. [24, 26] Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces. A function $f : X \to Y$ is closure-preserving (or continuous), if for all $A \in \wp(X)$, $f(cl_X(A)) \subseteq cl_Y(f(A))$.

Now, for the isotonic spaces, (X, Δ) and (X, ∇), it is obvious that $f(\nabla(A)) \subseteq f(\Delta(A))$, since, $\nabla(A) \subseteq \Delta(A)$, for any function $f : X \to X$ and for any $A \in \wp(X)$.

Further, if the function $f : (X, \Delta) \to (X, \nabla)$ is closure-preserving (or continuous), then, $f(\Delta(A)) \subseteq \nabla f(A)$, for any subset $A \in \wp(X)$. Thus, we have following:

Theorem 4.3. Let $f : (X, \Delta) \to (X, \nabla)$ be a closure-preserving function. Then, $f(\nabla(A)) \subseteq f(\Delta(A)) \subseteq \nabla(f(A))$, for all $A \in \wp(X)$.

We define homeomorphism between two isotonic spaces from [25]:

Definition 4.4. If (X, cl) and (Y, cl) are isotonic spaces and $f : (X, cl_X) \to (Y, cl_Y)$ is a bijection, then f is a homeomorphism if and only if $f(cl_X(A)) = cl_Y(f(A))$, for every $A \in \wp(X)$.

Corollary 4.5. Let $f : (X, \Delta) \to (X, \nabla)$ be a bijective closure-preserving function such that $\nabla f(A) \subseteq f(\Delta(A))$, for all $A \in \wp(X)$. Then, f is a homeomorphism.

Theorem 4.6. The identity function $i:(X,\nabla)\to(X,\Delta)$ is always a closure-preserving (or continuous) function.

Proof. We know that $i(\nabla(A))\subseteq i(\Delta(A))=\Delta(A)=\Delta(i(A))$.

Example 4.7. Let $X=\{a,b\}$, $\tau=\{\varnothing,\{a\},X\}$ and $\mathcal{I}=\{\varnothing,\{a\}\}$. Let $A=\{a\}$ and $i:(X,\Delta)\to(X,\nabla)$ be the identity function. Then, $i(A)=A$, $\Delta(A)=\{a\}$ and $\nabla(A)=\varnothing$. So, $i(\Delta(A))\nsubseteq\nabla(i(A))$. This example shows that the identity function $i:(X,\Delta)\to(X,\nabla)$ may not be a closure-preserving function.

Corollary 4.8. A closure-preserving bijective mapping $f:(X,\Delta)\to(X,\nabla)$ is homeomorphism, if and only if, $\nabla(f(A))\subseteq f(\Delta(A))$, for all $A\in\wp(X)$.

Proof. Suppose, $\nabla(f(A))\subseteq f(\Delta(A))$. Then, from the Corollary 4.5, f is a homeomorphism. Conversely, suppose $f:(X,\Delta)\to(X,\nabla)$ is a homeomorphism, then $\nabla(f(A))\subseteq f(\Delta(A))$ is obvious.

Definition 4.9. [26] A generalized closure space (X,cl) is a T_0-space if and only if for any $x,y\in X$ with $x\neq y$, there exists $N_x\in\mathcal{N}(x)$ (where $\mathcal{N}(x)=\{N\in\wp(X):x\in Int(N)\}$) such that $y\notin N_x$ or there exists $N_y\in\mathcal{N}(y)$ (where $\mathcal{N}(y)=\{N\in\wp(X):y\in Int(N)\}$) such that $x\notin N_y$.

Definition 4.10. [25] A generalized closure space (X,cl) is a T_1-space if, for any $x,y\in X$ with $x\neq y$, there exists $N'\in\mathcal{N}(x)$ and $N''\in\mathcal{N}(y)$ such that $x\notin N''$ and $y\notin N'$.

Definition 4.11. [25] A generalized closure space (X,cl) is a T_2-space if and only if, for all $x,y\in X$ with $x\neq y$, there exists $N'\in\mathcal{N}(x)$ and $N''\in\mathcal{N}(y)$ such that $N'\cap N''=\varnothing$.

Definition 4.12. [25] A space (X,cl) is a $T_{2\frac{1}{2}}$-space if and only if, for all $x,y\in X$ with $x\neq y$, there exists $N'\in\mathcal{N}(x)$ and $N''\in\mathcal{N}(y)$ such that $cl(N')\cap cl(N'')=\varnothing$.

Theorem 4.13. Let $f:(X,\Delta)\to(X,\nabla)$ be a bijective closure-preserving function such that $\nabla(f(A))\subseteq f(\Delta(A))$, for all $A\in\wp(X)$. Then, the followings hold:

1. (X,Δ) is a T_0-space, if and only if, (X,∇) is a T_0-space.
2. (X,Δ) is a T_1-space, if and only if, (X,∇) is a T_1-space.
3. (X,Δ) is a T_2-space, if and only if, (X,∇) is a T_2-space.
4. (X,Δ) is a $T_{2\frac{1}{2}}$-space, if and only if, (X,∇) is a $T_{2\frac{1}{2}}$-space.

Definition 4.14. Let (X,cl_X) and (Y,cl_Y) be two generalized closure spaces. A function $f:X\to Y$ is called anti closure-preserving if $cl_Y(f(A))\subseteq f(cl_X(A))$, for all $A\in\wp(X)$.

Existence of anti closure-preserving function:

Example 4.15. Let $X=\{a,b,c\}=Y$. Let us define $cl_X:\wp(X)\to\wp(X)$ by, $cl_X(\varnothing)=\varnothing$, $cl_X(\{a\})=\{a\}$, $cl_X(\{b\})=\{b\}$, $cl_X(\{c\})=\{c\}$, $cl_X(\{a,b\})=\{a,b\}$, $cl_X(\{a,c\})=\{a,b\}$, $cl_X(\{b,c\})=\{b,c\}$, $cl_X(X)=X$ and $cl_Y:\wp(X)\to\wp(X)$ by $cl_Y(\varnothing)=\varnothing$, $cl_Y(\{a\})=\{a,b\}$, $cl_Y(\{b\})=\{b,c\}$, $cl_Y(\{c\})=\{b,c\}$, $cl_Y(\{a,b\})=Y$, $cl_Y(\{a,c\})=Y$, $cl_Y(\{b,c\})=\{b,c\}$, $cl_Y(Y)=Y$.

Define $f:(Y,cl_Y)\to(X,cl_X)$ by $f(x)=x$. Then $cl_X(f(\varnothing))=\varnothing$, $cl_X(f(Y))=X$, $cl_X(f\{a\})=\{a\}, cl_X(f\{b\})=\{b\}, cl_X(f\{c\})=\{c\}, cl_X(f\{a,b\})=\{a,b\}=cl_X(f\{a,c\})$, $cl_X(f\{b,c\})=\{b,c\}$ and $f(cl_Y(\varnothing))=\varnothing$, $f(cl_Y(\{a\}))=\{a,b\}$, $f(cl_Y(\{b\}))=\{b,c\}$ $=f(cl_Y(\{c\}))=f(cl_Y(\{b,c\}))$, $f(cl_Y(\{a,b\}))=X=f(cl_Y(\{a,c\}))=f(cl_Y(Y))$.

Thus $cl_X(f(\varnothing))=f(cl_Y(\varnothing)), cl_X(f(Y))=f(cl_Y(Y)), cl_X(f\{a\})\subseteq f(cl_Y(\{a\}))$,

$cl_X(f\{b\}) \subseteq f(cl_Y(\{b\}))$, $cl_X(f\{c\}) \subseteq f(cl_Y(\{c\}))$, $cl_X(f\{a,b\}) \subseteq f(cl_Y(\{a,b\}))$,

$cl_X(f\{b,c\}) \subseteq f(cl_Y(\{b,c\}))$, $cl_X(f\{a,c\}) \subseteq f(cl_Y(\{a,c\}))$.

Thus we see that f is an anti closure-preserving function.

Note that the identity function $i:(X,\Delta) \to (Y,\nabla)$ is always an anti closure-preserving function, since, for all $A \subseteq X$, $\nabla(i(A)) = \nabla(A) \subseteq \Delta(A) = i(\Delta(A))$.

Remark 4.16. We can replace "$\nabla(f(A)) \subseteq f(\Delta(A))$" in Corollary 4.5, Corollary 4.8 and Theorem 4.13 by "f is an anti closure-preserving function".

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