Aksaray University
Journal of Science and Engineering

e-ISSN: 2587-1277
http://asujse.aksaray.edu.tr

Aksaray J. Sci. Eng.
Volume **8,** Issue **2,** pp. **114-121.**

Available online at DergiPark

Research Article

Sum and Weighted Sum Formulas for Fibonacci and Lucas Quaternions

Adnan KARATAŞ[1,*]

[1]*Pamukkale University, Department of Mathematics, Türkiye*

Keywords
Fibonacci Numbers, Lucas Numbers, Sum Formulas, Weighted Sum Formulas, Double Coefficient Fibonacci Numbers

Article information
Received: Oct 31, 2024
Revised: Nov 15, 2024
Accepted: Dec 27, 2024
Online: Dec 30, 2024

doi: **10.29002/asujse.1576933**

Abstract

Fibonacci and Lucas quaternions are deeply studied in the literature after its definition by Horadam in 1963. Binet formula, generation function, Cassini, Catalan, Honsberger, and other identities for these sequences studied in different quaternion algebra types like split quaternion algebra, dual quaternion algebra, etc. Additionally, some of the generalizations for Fibonacci and Lucas quaternions are studied like k-Fibonacci quaternions, k-Lucas quaternions, Horadam quaternions, and more. However, despite being researched intensively, the summation formulas or weighted summation formulas for these sequences are not examined thoroughly. In this study, after briefly summarizing the literature, we focused on the summation and the weighted summation formulas for the Fibonacci and the Lucas quaternions. In addition, we provided generalized weighted summation formulas for the Fibonacci and Lucas quaternions. Moreover, we calculated generalized weighted summation formulas for the double coefficient Fibonacci and double coefficient Lucas quaternions which have two Fibonacci and Lucas coefficient in every unit of the quaternion, respectively.

1. Introduction

The quaternion algebra has a well-known history of discovery. In 1844, W. R. Hamilton engraved the idea to the Broom bridge in Dublin, Ireland. The definition of quaternions is the starting point of many different types of algebras, like split quaternions, octonions or sedenions.

Apart from the quaternion algebra's role in pure mathematics and physics, it has an indisputable place in three-dimensional rotations and computational geometry and graphics [1,2]. In [1], authors searched the literature about the applications of quaternion algebra and geometric algebra between 1995 and 2020. Besides, in [2], authors provide a literature review about image processing methods with color using quaternions.

We only provide basic definitions for the quaternion algebra as a reminder. Throughout the study we use real quaternion algebra which has real numbers as a coefficient. A quaternion q is defined as

$$q = q_0 + \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3$$

where the $q_0, q_1, q_2,$ and q_3 are the real numbers.

Addition of the quaternions is component-wise and multiplication operation of two quaternions can be made employing the Table 1.

Table 1. The multiplication table of quaternion units

$*$	i	j	k
i	-1	k	$-$j
j	$-$k	-1	i
k	j	$-$i	-1

As it can be seen from the table quaternion algebra is a non-commutative but associative four-dimensional algebra.

The norm form for the quaternion algebra is defined as follows.

*Corresponding Author: ▪e-mail:adnank@pau.edu.tr 0000-0003-3652-5354

$$N(q) = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

In addition, it can be observed that the quaternions have non-degenerate or non-singular norm form.

Another aspect of the quaternion algebra worth mentioning is its matrix representation. In [3], it is stated that quaternion matrix representation has advantage over usage of orthogonal matrices. To provide more insight, the rotations with orthogonal matrices require 9 slots for storage, but the quaternion matrix representation needs only 4. Additionally, for composition of rotations, the orthogonal matrices require 27 multiplications and 18 additions, whereas these values are 16 and 12, if one uses quaternion matrix representation. It can be clearly seen that, usage of the quaternion matrix representation leads to less memory consumption and has fewer operations in comparison to the orthogonal matrices.

The mentioned rotation matrix using quaternions is given below. A rotation in $\mathbb{R}^3$ can be made by following matrix M with $q = q_0 + \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3$ and $N(q) = 1$.

$$M = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix}$$

It is obvious that the M is orthogonal, with $\det(M) = 1$.

In the following section, we state the Fibonacci and Lucas quaternions, after giving necessary identities we present our main theorems. In the last section, we summarize the work and provide future study ideas.

2. Summation Formulas for Fibonacci and Lucas Quaternions

The Fibonacci sequence has initial values $f_0 = 0$ and $f_1 = 1$. The recurrence relation for the sequence is

$$f_{n+2} = f_{n+1} + f_n \tag{1}$$

The Lucas sequence has initial values $l_0 = 2$ and $l_1 = 1$. Having the same form of the recurrence relation with Fibonacci sequence, the Lucas sequence usually studied with the Fibonacci sequence.

The First definition of Fibonacci quaternion was made in 1963 by A. F. Horadam [4]. After that, in [5] the author provided many identities for the sequence. In 2012, the author gave Binet and generating function for Fibonacci numbers with other identities [6]. Some papers include summation formulas for Fibonacci and Lucas quaternions or for their generalizations, see [7-9]. Other studies needed to be mentioned can be given as [10-18].

Sum and weighted sum formulas for Fibonacci and Lucas numbers have applications to graph theory. For more information see, [19,20].

Before giving sum formulas for Fibonacci and Lucas quaternions we restate the definitions of them. A quaternion with Fibonacci coefficients is defined as follows.

$$Q_n = f_n + \mathrm{i}f_{n+1} + \mathrm{j}f_{n+2} + \mathrm{k}f_{n+3} \tag{2}$$

The recurrence relation is

$$Q_{n+2} = Q_{n+1} + Q_n$$

where $Q_0 = \mathrm{i} + \mathrm{j} + \mathrm{k}2$ and $Q_1 = 1 + \mathrm{i} + \mathrm{j}\,2 + \mathrm{k}3$.

The Lucas quaternion and its initial values are defined as follows

$$QL_n = l_n + \mathrm{i}l_{n+1} + \mathrm{j}l_{n+2} + \mathrm{k}l_{n+3} \tag{3}$$

$QL_0 = 2 + \mathrm{i} + \mathrm{j}3 + \mathrm{k}4$ and $QL_1 = 1 + \mathrm{i}3 + \mathrm{j}4 + \mathrm{k}7$.

In order to employ and check the compatibility of formulas, we want to mention several identities from the references, [6,21].

$$\sum_{i=1}^{n} f_i = f_{n+2} - 1,$$

$$\sum_{i=1}^{n} l_i = l_{n+2} - 3,$$

$$\sum_{i=1}^{n} i f_i = n f_{n+2} - f_{n+3} + 2,$$

$$\sum_{i=1}^{n} i l_i = n l_{n+2} - l_{n+3} + 4,$$

$$\sum_{i=1}^{n} (n - i + 1) f_i = f_{n+4} - n - 3,$$

$$\sum_{i=1}^{n} (n - i + 1) l_i = l_{n+4} - 3n - 7,$$

$$\sum_{i=1}^{n} i f_i^2 = n f_n f_{n+1} - f_n^2 + \gamma$$

where $\gamma = 1$ if n is odd, else $\gamma = 0$.

$$\sum_{i=1}^{n} i l_i^2 = n l_n l_{n+1} - l_n^2 + \nu$$

where $\nu = -1$ if n is odd, else $\nu = 4$.

$$\sum_{i=1}^{n} Q_i = Q_{n+2} - Q_2,$$

$$\sum_{i=1}^{n} Q_{2i} = Q_{2n+1} - Q_1,$$

$$\sum_{i=1}^{n} Q_{2i-1} = Q_{2n} - Q_{0,}$$

In the following theorem, we provide summation formulas for Lucas quaternions.

Theorem 1. For Lucas quaternions following equations hold true.

$$\sum_{i=1}^{n} QL_i = QL_{n+2} - QL_2, \tag{4}$$

$$\sum_{i=1}^{n} QL_{2i} = QL_{2n+1} - QL_1, \tag{5}$$

$$\sum_{i=1}^{n} QL_{2i-1} = QL_{2n} - QL_0. \tag{6}$$

Proof. We only prove the Equation (5) for brevity. Using induction, for $n = 1$

$$QL_2 = QL_3 - QL_1$$

is true. Now assume it is true for an arbitrary positive integer k. Then

$$\sum_{i=1}^{k} QL_{2i} = QL_{2k+1} - QL_1.$$

Adding QL_{2k+2} to the both sides leads to

$$\sum_{i=1}^{k+1} QL_{2i} = QL_{2k+2} + QL_{2k+1} - QL_1 = QL_{2k+3} - QL_1.$$

In the next theorem, we present a weighted sum formula for the Fibonacci quaternions.

Theorem 2. For positive integer n, the following equations hold true.

$$\sum_{i=1}^{n} iQ_i = nQ_{n+2} - Q_{n+3} + Q_{3,} \quad (7)$$

$$\sum_{i=1}^{n} (n-i+1)Q_i = Q_{n+4} - nQ_2 - Q_4. \quad (8)$$

Proof. The proof of the Equation (7) is as follows. For $n = 1$,

$$Q_1 = Q_3 - Q_4 + Q_3$$

is true. For induction, assuming the equation is valid for arbitrary positive integer k

$$\sum_{i=1}^{k} iQ_i = kQ_{k+2} - Q_{k+3} + Q_3.$$

Adding necessary component $(k+1)Q_{k+1}$to the both sides gives us

$$\sum_{i=1}^{k+1} iQ_i = (k+1)Q_{k+1} + kQ_{k+2} - Q_{k+3} + Q_3 = (k+1)Q_{k+3} - Q_{k+4} + Q_3.$$

The other equation can be proven similarly.

The Lucas counterparts of the Theorem 2 are given below.

$$\sum_{i=1}^{n} iQL_i = nQL_{n+2} - QL_{n+3} + QL_3, \quad (9)$$

$$\sum_{i=1}^{n} (n-i+1)QL_i = QL_{n+4} - nQL_2 - QL_4. \quad (10)$$

In the following theorem, we present generalized weighted and generalized reversed weighted sum formulas for Fibonacci quaternions.

Theorem 3. For Fibonacci quaternions, the following generalized weighted and generalized reversed weighted sum formulas hold true.

$$\sum_{i=1}^{n} [a + (i-1)d]Q_i = [a + (n-1)d]Q_{n+2} - dQ_{n+3} + (d-a)Q_2 + dQ_3, \quad (11)$$

$$\sum_{i=1}^{n} [a + (n-i)d]Q_i = aQ_{n+2} + dQ_{n+3} - (a+nd)Q_2. \quad (12)$$

Proof. In order to prove the theorem, we start with the Equation (11).

$$\sum_{i=1}^{n} [a + (i-1)d]Q_i = a\sum_{i=1}^{n} Q_i + d\sum_{i=1}^{n} iQ_i - d\sum_{i=1}^{n} Q_i.$$

Using Equation (2) and Equation (7) in the summation formula above, we obtain the desired result. For Equation (12), we use the fact that

$$\sum_{i=1}^{n}[a+(i-1)d]Q_i+\sum_{i=1}^{n}[a+(n-i)d]Q_i=$$

$$[2a+(n-1)d]\sum_{i=1}^{n}Q_i=[2a+(n-1)d](Q_{n+2}-Q_2)$$

The Equation (12), can be calculated from

$$\sum_{i=1}^{n}[a+(n-i)d]Q_i=[2a+(n-1)d]\sum_{i=1}^{n}Q_i-\sum_{i=1}^{n}[a+(i-1)d]Q_i.$$

After computing equations, we obtain the following result.

$$\sum_{i=1}^{n}[a+(n-i)d]Q_i=aQ_{n+2}+dQ_{n+3}-(a+nd)Q_2-dQ_3.$$

In the Equation (11), choosing $a=d=1$ reveals the Equation (7). In addition, for $a=d=1$, the Equation (12) is equal to the Equation (8).

For the Lucas counterpart of these equations, similar steps can be taken and the following equations can be obtained.

$$\sum_{i=1}^{n}[a+(i-1)d]QL_i=[a+(n-1)d]QL_{n+2}-dL_{n+3}+(d-a)QL_2+dQL_3, \tag{13}$$

$$\sum_{i=1}^{n}[a+(n-i)d]QL_i=aQL_{n+2}+dQL_{n+3}-(a+nd)QL_2-dQL_3. \tag{14}$$

To check compatibility with earlier results, one can take a and d to be 1 in Equation (13) and Equation (14) which leads to the Equation (9) and Equation (10), respectively.

For the following theorems, we use the following double coefficient quaternion notation.

$$Q_{(n,m)}=f_nf_m+\mathrm{i}f_{n+1}f_{m+1}+\mathrm{j}f_{n+2}f_{m+2}+\mathrm{k}f_{n+3}f_{m+3}. \tag{15}$$

Employing the notation above, we state the following theorem.

Theorem 4. For positive integer n, the following equations hold true.

$$\sum_{i=1}^{n}Q_{(i,i)}=Q_{(n,n+1)}-A \tag{16}$$

where $A=(\mathrm{i}+\mathrm{j}2+\mathrm{k}6)$.

$$\sum_{i=1}^{n}iQ_{(i,i)}=nQ_{(n,n+1)}-Q_{(n,n)}+\Gamma \tag{17}$$

where $\Gamma=\begin{cases}1+\mathrm{j}2+\mathrm{k}3, & if\ n\ is\ odd\\ \mathrm{i}+\mathrm{j}+\mathrm{k}4, & otherwise\end{cases}$.

Proof. The proof of the Equation (16) and Equation (17) can be made by Binet formula of Fibonacci quaternions or by using identities. But for the length of the paper, we employ the induction method.

$$Q_{(n,n)}=f_nf_n+\mathrm{i}f_{n+1}f_{n+1}+\mathrm{j}f_{n+2}f_{n+2}+\mathrm{k}f_{n+3}f_{n+3}.$$

First, we check the case $n=1$

$$Q_{(1,1)}=Q_{(1,2)}-A.$$

Second, we assume the Equation (16) is true for $n=k$ and add $Q_{(k+1,k+1)}$ to the equation.

$$\sum_{i=1}^{k}Q_{(i,i)}+Q_{(k+1,k+1)}=Q_{(k,k+1)}-A+Q_{(k+1,k+1)}.$$

Finally, using the equality, $f_k f_{k+1} + f_{k+1} f_{k+1} = f_{k+1} f_{k+2}$, gives us the desired result.

The Lucas counterparts of the Theorem 4 are given below.

$$\sum_{i=1}^{n} QL_{(i,i)} = QL_{(n,n+1)} - B \tag{18}$$

where $B = -(2 + \mathrm{i}3 + \mathrm{j}12 + \mathrm{k}28)$.

$$\sum_{i=1}^{n} iQL_{(i,i)} = nQL_{(n,n+1)} - QL_{(n,n)} + N \tag{19}$$

where $N = \begin{cases} -1 + \mathrm{i}6 + \mathrm{j}4 + \mathrm{k}21, & if\ n\ is\ odd \\ 4 + \mathrm{i}1 + \mathrm{j}9 + \mathrm{k}16, & otherwise \end{cases}$.

The generalized form of the Theorem 4 is given in the following theorem.

Theorem 5. For double coefficient Fibonacci quaternions, weighted and reversed weighted sum formulas are given, respectively.

$$\sum_{i=1}^{n} [a + (i-1)d] Q_{(i,i)} = [a + (n-1)d] Q_{(n,n+1)} - dQ_{(n,n)} + (d-a)A + d\Gamma \tag{20}$$

where $A = (\mathrm{i} + \mathrm{j}2 + \mathrm{k}6)$.

$$\sum_{i=1}^{n} [a + (n-i)d] Q_{(i,i)} = aQ_{(n,n+1)} + dQ_{(n,n)} - (a+nd)A - d\Gamma \tag{21}$$

where $\Gamma = \begin{cases} 1 + \mathrm{j}2 + \mathrm{k}3, & if\ n\ is\ odd \\ \mathrm{i} + \mathrm{j} + \mathrm{k}4, & otherwise \end{cases}$.

Proof. To prove the theorem, firstly, we use properties of sum and earlier results.

$$\sum_{i=1}^{n} [a + (i-1)d] Q_{(i,i)} = a \sum_{i=1}^{n} Q_{(i,i)} + d \sum_{i=1}^{n} iQ_{(i,i)} - d \sum_{i=1}^{n} Q_{(i,i)},$$

$$LHS = a\big(Q_{(n,n+1)} - A\big) + d\big(nQ_{(n,n+1)} - Q_{(n,n)} + \Gamma\big) - d\big(Q_{(n,n+1)} - A\big),$$

$$LHS = [a + (n-1)d] Q_{(n,n+1)} - dQ_{(n,n)} + (d-a)A + d\Gamma$$

which proves the Equation (20). Secondly, to calculate Equation (21), we use the following

$$\sum_{i=1}^{n} [a + (i-1)d] Q_{(i,i)} + \sum_{i=1}^{n} [a + (n-i)d] Q_{(i,i)}$$

$$= [2a + (n-1)d] \sum_{i=1}^{n} Q_{(i,i)} = [2a + (n-1)d] Q_{(n,n+1)} - A.$$

Finally, employing the equation above, we can compute the Equation (21).

$$\sum_{i=1}^{n} [a + (n-i)d] Q_{(i,i)} = [2a + (n-1)d] \sum_{i=1}^{n} Q_{(i,i)} - \sum_{i=1}^{n} [a + (i-1)d] Q_{(i,i)}$$

$$\sum_{i=1}^{n} [a + (n-i)d] Q_{(i,i)} = aQ_{(n,n+1)} + dQ_{(n,n)} - (a+nd)A - d\Gamma.$$

Note that, in the Equation (20) choosing $a = d = 1$ gives the Equation (17).

The double coefficient Lucas counter-part of generalized weighted sum and generalized reversed weighted sum formulas are given below.

$$\sum_{i=1}^{n}[a+(i-1)d]QL_{(i,i)}=[a+(n-1)d]QL_{(n,n+1)}-dQL_{(n,n)}+(d-a)B+dN, \quad (22)$$

$$\sum_{i=1}^{n}[a+(n-i)d]QL_{(i,i)}=aQL_{(n,n+1)}+dQL_{(n,n)}-(a+nd)B-dN \quad (23)$$

where B and N is from Equations (18) and Equation (19), respectively. In addition, choosing $a=1$ and $d=1$ in Equation (22) gives the Equation (19) which shows the compatibility of the result.

3.Conclusion

As a result, we focused on Fibonacci and Lucas quaternions which are researched extensively in the literature. However the summation formulas, especially weighted summation formulas, are not studied. We provided numerous summation formulas for both Fibonacci and Lucas quaternions. Moreover, we presented generalized weighted and generalized reversed weighted sum formulas which are compatible with other summation results. In the future studies, summation and weighted summation formulas for different sequences can be studied.

References

[1] Bayro-Corrochano E., **(2021).** *A survey on quaternion algebra and geometric algebra applications in engineering and computer science 1995–2020*. IEEE Access, 9: 104326–104355.

[2] Huang C., Li J., Gao G., **(2023).** *Review of quaternion-based color image processing methods*. Mathematics, 11: 2056.

[3] Salamin E., **(1979).** *Application of quaternions to computation with rotations*. Tech. rep., Working Paper.

[4] Horadam A.F., **(1963).** *Complex Fibonacci numbers and Fibonacci quaternions*. The American Mathematical Monthly, 70: 289–291.

[5] Iyer M.R., **(1969).** *Some results on Fibonacci quaternions*. The Fibonacci Quarterly, 7: 201–210.

[6] Halici S., **(2012).** *On Fibonacci quaternions*. Adv. Appl. Clifford Algebras, 22: 321–327.

[7] Polatli E., Kesim S., **(2015).** *On quaternions with generalized Fibonacci and Lucas number components.* Advances in Difference Equations, 1: 1–8.

[8] Ramirez J.L., **(2015).** *Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions*. Analele stiintifice ale Universitatii Ovidius Constanta. Seria Matematica, 23: 201–212.

[9] Akyigit M., Hüda Kösal H., Tosun M., **(2014).** *Fibonacci generalized quaternions*. Advances in Applied Clifford Algebras, 24: 631–641.

[10] Bitim B.D., **(2019).** *Some identities of Fibonacci and Lucas quaternions by quaternion matrices*. Düzce Üniversitesi Bilim ve Teknoloji Dergisi, 7: 606–615.

[11] Cerda-Morales G., **(2017).** *Some properties of horadam quaternions*. arXiv preprint, arXiv:1707.05918.

[12] Flaut C., Savin D., **(2015).** *Quaternion algebras and generalized Fibonacci- Lucas quaternions*. Advances in Applied Clifford Algebras, 25: 853–862.

[13] Halici S., Karataş A., **(2017).** *Some matrix representations of Fibonacci quaternions and octonions*. Advances in Applied Clifford Algebras, 27: 1233–1242.

[14] Irmak N., **(2020).** *More identities for fibonacci and lucas quaternions*. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69: 369–375.

[15] Köme S., Köme C., Yazlık Y., **(2019).** *Modified generalized Fibonacci and Lucas quaternions*. Journal of Science and Arts, 19: 49-60.

[16] Patel B.K., Ray P.K., **(2019).** *On the properties of (p, q)-Fibonacci and (p, q)-Lucas quaternions*. Mathematical Reports, 21: 15–25.

[17] Polatlı E., **(2016).** *A generalization of Fibonacci and Lucas quaternions*. Advances in Applied Clifford Algebras, 26: 719–730.

[18] Swamy M., **(1973).** *On generalized Fibonacci quaternions*. The Fibonacci Quarterly, 11: 547–550.

[19] Tasci D., Buyukkose S., Kizilirmak G.O., Sevgi E., **(2020).** *Properties of Lucas-sum graph*. Journal of Science and Arts, 20: 313–316.

[20] Kilic E., Stakhov A., **(2009).** *On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices*. Chaos, Solitons & Fractals, 40: 2210–2221.

[21] Koshy T., **(2019)**, *Fibonacci and Lucas Numbers with Applications*. John Wiley & Sons.