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Research Article

Determination of the Characteristic Class Functions of the Weyl Group of type *G*2

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Abstract

In this paper, our main objective is to combinatorially obtain all characteristic class functions of the Weyl group of type G_2 with the help of the structure of generalized descent algebra including Solomon's descent algebra and Mantaci-Reutenauer algebra. We also give an interpretation of the character table of the Weyl group of type G_2 in terms of the permutation characters defined on the set of cosets belonging to each of its some special subgroups.

1. Introduction

Let W be a finite Weyl group with the canonical generating set S. The collection of all reflections in W is $\mathrm{Ref}(W) = \bigcup_{w \in W} wSw^{-1}$. The length function of W depending on S is defined as $l : W \to \mathbb{N}$, $l(w) :=$ the number of simple reflections in the irreducible expression of w. Now let us consider the sums of the form

$$\sum_{w \in W} \chi(w) q^{l(w)} \tag{1}$$

where χ is the characteristic class function of W associated with the conjugacy class K_χ. In the case of χ being a one dimensional character, the sum in Eq. (1) was investigated in detail by Reiner Ref. [1]. When χ is the trivial character, the corresponding sum of Eq. (1) is the Poincaré polynomial of W, which has an elegant product formula (see Ref. [2]). The definition of the characteristic class function yields the following relation

$$\sum_{w \in W} \chi(w) q^{l(w)} = \sum_{w \in \mathrm{K}_\chi} q^{l(w)}. \tag{2}$$

If we take $q = 1$ in the Eq. (2), then we obtain the cardinality of the conjugacy class K_χ. Substituting $q = -1$ in the previous equation and using the inner product of characters, we get

$$\sum_{w \in W} \chi(w)(-1)^{l(w)} = \sum_{w \in \mathrm{K}_\chi} (-1)^{l(w)} = |W| \langle \chi, \varepsilon \rangle \tag{3}$$

where ε denotes the sign character of W defined by $\varepsilon(w) = (-1)^{l(w)}$ for any w element of W. Since the sign character takes a constant value on each conjugacy class, then we write

$$\sum_{w \in W} \chi(w)(-1)^{l(w)} = (-1)^{l(k_\chi)} |\mathrm{K}_\chi|, \tag{4}$$

where k_χ is independent of the choice of $k_\chi \in \mathrm{K}_\chi$.

The inner product $\langle \chi, \varepsilon \rangle$ is also equal to the coefficient of the sign character ε in the expression of the characteristic class function χ in relation to irreducible characters of W. Taking the Eqs. (3) and (4) into consideration, for any finite Weyl group W we deduce that the sign character ε appears exactly

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$$\frac{(-1)^{l(k_\chi)}}{|W|}|\mathrm{K}_\chi|$$

times in the expression of a characteristic class function χ in terms of irreducible characters of W. Since it is well-known that the collection of the elements $\sum_{w\in\mathrm{K}_\chi} w$ for all conjugacy classes K_χ in W is a basis for the centre of group algebra $\mathbb{Q}W$, the relation in the Eq. (3) actually determines the action of the sign character ε on the centre of group algebra $\mathbb{Q}W$ when the sign character ε is extended to group algebra by linearity.

The problem is here that how to practically calculate the sizes of conjugacy classes in the Eq. (4). In the case of the hyperoctahedral group W_n all characteristic class functions were more explicitly constructed by using the epimorphism between Mantaci-Reutenauer algebra $\mathrm{MR}(W_n)$ established in Ref. [3] and $\mathbb{Q}\,\mathrm{Irr}W_n$ and an effective formula for determining the size of any conjugacy class of the group W_n was derived in Ref. [4]. From Corollary 19 in Ref. [4], for any double partition λ of n we write the size of K_λ as $|W_n|\sum_{\mu\in\mathrm{DP}(n)} u_{\lambda,\mu} = |\mathrm{K}_\lambda|$, where $\mathrm{DP}(n)$ and K_λ represent the collections of all double partitions of n and the conjugacy class corresponding to λ, respectively.

In section 2, we construct all the characteristic class functions of the Weyl group of type G_2 depending on the structure of *generalized descent algebra* described in Ref. [5]. Generalized descent algebra covers both the classical Solomon's descent algebras constructed in Ref. [6] and Mantaci-Reutenauer algebra.

2. Construction of the Characteristic Class Functions of the Weyl Group of Type G_2

We consider the Weyl group $W = D_{12} = \{s,t : s^2 = t^2 = (st)^6 = 1\}$ of type G_2. The conjugacy classes and all irreducible characters of this group are $\{1\}$, $\{s,tst,ststs\}$, $\{t,sts,tstst\}$, $\{st,ts\}$, $\{stst,tsts\}$, $\{(st)^3\}$ and $1_{D_{12}}, \varepsilon, \delta, \varepsilon\delta, \psi_1, \psi_2$, respectively. We note here that all linear characters of the group are exactly the characters $1_{D_{12}}, \varepsilon, \delta, \varepsilon\delta$ and that δ is defined by the relations $\delta(s) = -\delta(t) = 1$. Both characters ψ_1 and ψ_2 are two dimensional and their values on the representatives of the conjugacy classes appear in Table 1 complete character table of D_{12} extracted from Ref. [7]:

Table 1. The character table of D_{12}

	1	s	t	st	$(st)^2$	$(st)^3$
$1_{D_{12}}$	1	1	1	1	1	1
ε	1	-1	-1	1	1	1
δ	1	1	-1	-1	1	-1
$\varepsilon\delta$	1	-1	1	-1	1	-1
ψ_1	2	0	0	1	-1	-2
ψ_2	2	0	0	-1	-1	2

Fix the reflection subset $B = \{s,t,sts,tstst\}$ of $\mathrm{Ref}(D_{12})$, where $\mathrm{Ref}(D_{12}) = \{s,t,sts,tst,ststs,tstst\}$ is the set of all reflections in D_{12}. In [5], Hohlweg constructed the generalized descent algebra $\sum_{\mathrm{P}_0(B)}(D_{12})$ which is spanned by the elements x_A, where $A \in \mathrm{P}_0(B) = \{\varnothing,\{s\},\{t\},\{sts\},\{tstst\},\{s,tstst\},\{t,sts,tstst\},B\}$ such that $W_A \cap B = A$, W_A is a reflection subgroup of D_{12} generated by A, X_A is the set of distinguished coset representatives of W_A in D_{12} and $x_A = \sum_{w\in X_A} w$. In addition, he proved that the map

$\Phi_B : \sum_{\mathrm{P}_0(B)}(D_{12}) \to \square\, \mathrm{Irr}(D_{12}), x_A \mapsto \mathrm{Ind}_{W_A}^{D_{12}} 1_{W_A}$ is a linear map, where 1_{W_A} and $\mathrm{Ind}_{W_A}^{D_{12}} 1_{W_A}$ stand for the trivial character of W_A and the permutation character of D_{12} on the cosets of W_A. Furthermore, the map Φ_B is surjective and not a morphism of algebras, where $\mathrm{Ind}_{W_A}^{D_{12}} 1_{W_A}$ is the permutation character of W_A in D_{12}. Using the table in the proof of Proposition 2.3 in Ref. [5], one can derive the expression of the permutation characters $\Phi_B(x_A)$ in terms of irreducible characters of D_{12}, which is given in Table 2. The regular character of D_{12} is denoted by χ_{reg}.

Table 2. Expression of the permutation characters $\Phi_B(x_A)$ in terms of irreducible characters

$\Phi_B(x_B)$	$1_{D_{12}}$
$\Phi_B(x_{\{s\}})$	$1_{D_{12}} + \delta + \psi_1 + \psi_2$
$\Phi_B(x_{\{t\}})$	$1_{D_{12}} + \varepsilon\delta + \psi_1 + \psi_2$
$\Phi_B(x_{\{sts\}})$	$1_{D_{12}} + \varepsilon\delta + \psi_1 + \psi_2$
$\Phi_B(x_{\{tstst\}})$	$1_{D_{12}} + \varepsilon\delta + \psi_1 + \psi_2$
$\Phi_B(x_{\{s,tstst\}})$	$1_{D_{12}} + \psi_2$
$\Phi_B(x_{\{t,sts,tstst\}})$	$1_{D_{12}} + \varepsilon\delta$
$\Phi_B(x_\varnothing)$	χ_{reg}

Table 3. Interpretation of the character table of D_{12} in terms of the permutation characters $\Phi_B(x_A)$

	st	$(st)^2$	$(st)^3$	t	s	1
$\Phi_B(x_B)$	1	1	1	1	1	1
$\Phi_B(x_{\{t,sts,tstst\}})$	0	2	0	2	0	2
$\Phi_B(x_{\{s,tstst\}})$	0	0	3	1	1	3
$\Phi_B(x_{\{t\}})$	0	0	0	2	0	6
$\Phi_B(x_{\{s\}})$	0	0	0	0	2	6
$\Phi_B(x_\varnothing)$	0	0	0	0	0	12

The values of the characters $\Phi_B(x_A)$ on all conjugacy classes of D_{12} are expressed as an upper triangular matrix with positive diagonal entries as in the Table 3. This is actually an interpretation of the character table of the group D_{12} according to permutation characters $\Phi_B(x_A), A \in \mathrm{P}_0(B)$.

Taking the inverse of the above matrix, we get

$$\begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{3} & \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} & 0 & \frac{-1}{2} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \frac{-1}{6} & \frac{-1}{6} & \frac{1}{12} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{-1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12} \end{pmatrix} \tag{5}$$

and thus determine the all characteristic class functions in terms of the permutation characters $\Phi_B(x_A)$ as follows:

$$\square e_{st} = \Phi_B(x_B) - \frac{1}{2}\Phi_B(x_{\{t,sts,tstst\}}) - \frac{1}{3}\Phi_B(x_{\{s,tstst\}}) + \frac{1}{6}\Phi_B(x_{\{t\}}) - \frac{1}{3}\Phi_B(x_{\{s\}}) + \frac{1}{6}\Phi_B(x_\varnothing)$$

$$\square e_{(st)^2} = \frac{1}{2}\Phi_B(x_{\{t,sts,tstst\}}) - \frac{1}{2}\Phi_B(x_{\{t\}}) + \frac{1}{6}\Phi_B(x_\varnothing)$$

$$\square e_{(st)^3} = \frac{1}{3}\Phi_B(x_{\{s,tstst\}}) - \frac{1}{6}\Phi_B(x_{\{t\}}) - \frac{1}{6}\Phi_B(x_{\{s\}}) + \frac{1}{12}\Phi_B(x_\varnothing)$$

- $e_t = \frac{1}{2}\Phi_B(x_{\{t\}}) - \frac{1}{4}\Phi_B(x_\varnothing)$
- $e_s = \frac{1}{2}\Phi_B(x_{\{s\}}) - \frac{1}{4}\Phi_B(x_\varnothing)$
- $e_1 = \frac{1}{12}\Phi_B(x_\varnothing)$

All characteristic class functions of D_{12} are also both a basis and a family of orthogonal primitive idempotents of the algebra $\mathbb{C}\,\mathrm{Irr}(D_{12})$. Because of the fact that each $e_\lambda,\ \lambda \in \left\{1, s, t, st, (st)^2, (st)^3\right\}$ satisfies the relation $< e_\lambda, \varepsilon > = \frac{(-1)^{l(k_\lambda)}|K_\lambda|}{12}$, one can from the Eq. (5) see that the coefficient of the sign character ε in the expression of each characteristic class function $e_\lambda,\ \lambda \in \left\{1, s, t, st, (st)^2, (st)^3\right\}$ in terms of irreducible characters of D_{12}, that is $\frac{(-1)^{l(k_\lambda)}|K_\lambda|}{12}$, is displayed in the last column.

We also observe that for any $A \in \mathrm{P}_0(B)$ the relation

$$< \Phi_B(x_A), \varepsilon > = \begin{cases} \frac{1}{12}, & A = \varnothing \\ 0, & A \neq \varnothing \end{cases}$$

is satisfied by virtue of the definition of iner product of characters. Moreover, we can practically find out the size of each conjugacy class of the group D_{12} in the sense of Corollary 19 in Ref. [4] when using the inverse matrix in the Eq. (5).

3. Conclusion

Descent algebras of finite Coxeter groups play a very important role in the representation theories of these groups and in explaining the structures of these groups. In Ref. [5], Hohlweg introduced the generalized descent algebras covering both Solomon's descent algebras and Mantaci-Reutenauer algebra. Then Hohlweg showed that the images of the all basis elements of the generalized descent algebra $\sum_{\mathrm{P}_0(B)}(D_{12})$ associated with Weyl group D_{12} of type G_2 under the $\Phi_B : \sum_{\mathrm{P}_0(B)}(D_{12}) \to \mathbb{C}\,\mathrm{Irr}(D_{12}), x_A \mapsto \mathrm{Ind}_{W_A}^{D_{12}} 1_{W_A}$ surjective linear map are equal to the irreducible characters of D_{12}.

Inspiring by the facts above, in this study, we expressed the permutation characters $\Phi_B(x_A)$ of the reflection subgroups W_A, $A \in \mathrm{P}_0(B)$ in D_{12} as a linear combination of the irreducible characters of the group, after that we reinterpreted the character table of the group in terms of these permutation characters. Then, we constructed all the characteristic class functions of the group using this new character table, which is in the form of an upper triangular matrix with all positive diagonal terms. Furthermore, having obtained all characteristic class functions of D_{12}, we determined the coefficient of the sign character ε in the expression of each characteristic class function in terms of irreducible characters of the group. Actually, the set of the characteristic class functions may be thinking of as a collection of orthogonal primitive idempotents of the semisimple algebra $\mathbb{C}\,\mathrm{Irr}(D_{12})$ and may be also considered as a basis of Burnside algebra $B(D_{12})$ of D_{12}.

Based on the characteristic class functions of D_{12} and the surjectivity of the linear map Φ_B, a family of orthogonal primitive idempotents of the generalized descent algebra $\sum_{\mathrm{P}_0(B)}(D_{12})$ may be constructed in future studies.

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