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On the Construction of the Surface Family with a Common Involute Geodesic

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Abstract

In this study, we produce a surface family possessing an involute of a given curve as a geodesic. We find necessary and sufficient conditions for the given curve such that its involute is a geodesic on any member of the surface family. Also, we present important results for ruled and developable surfaces. Finally, we present two examples to support our results.

Keywords: *Involute curve; Geodesic curve; Frenet frame; Surface family.*
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1. Introduction and Preliminaries

Involute curve was first discovered by Huygens when he was trying to make a more accurate clock. Involute curves have a wide range of mechanical engineering applications like involute gear teeth, centrifugal casing design, etc. The profiles of gear teeth are usually involute curves. This is the best form for keeping the teeth in contact, while minimizing wear and backlash $(Fig.1)$.

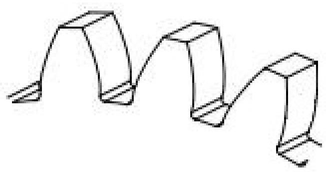

Fig. 1. Involute gear with involute teeth

The concept of family of surfaces having a given characteristic curve was first introduced by Wang et al. [2] in Euclidean 3-space. Kasap et al. [3] studied some surfaces using generalized marching-scale functions. Also, surfaces with common geodesic in Minkowski 3-space have been the subject of many studies [4, 5, 6, 7]. Bayram et al. [8] studied parametric surfaces which possess a given curve as a common asymptotic. Ergün et al. [10] constructed a surface pencil from a given spacelike (timelike) line of curvature in Minkowski 3-space. Recently Bayram and Bilici [9] expressed a surface family with a common involute asymptotic curve. In 2021, Bilici and Bayram [15] provided a parameterization to construct a surface family with a common involute line of curvature. For some recent work inspired by the involute-evolute curve pair, see [16, 17, 18, 19, 20, 21, 22, 23, 24]. In this paper, we give the necessary and sufficient condition for a given curve such that its involute is both isoparametric and geodesic on a parametric surface. Furthermore, we obtain some important results for ruled surfaces. Finally, we illustrate the method with two examples. Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a unit speed parametric curve, α' denotes the derivative of α with respect to arc lenght parameter s and we assume that α is a regular curve with $\alpha''(s) \neq 0$, where $s \in [n_1, n_2] \subset I$. Let $\{V_1(s), V_2(s), V_3(s)\}$ be the Frenet frame of α at the point $\alpha(s)$, where $V_1(s) = \alpha'(s)$, $V_2(s) = \frac{\alpha''}{\|\alpha''\|}$ and $V_3(s) = V_1(s) \times V_2(s)$ are the unit tangent, principal normal, and binormal vectors of the curve α, respectively. Derivative formulas of the Frenet frame are governed by the relations

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$$\frac{d}{ds}\begin{pmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{pmatrix},$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle V_3'(s), V_2(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [11].

Let α and β be two curves such that β intersects the tangents of α orthogonally. Then β is called an *involute* of α. An involute of a curve α with arc length s is given by

$$\beta(s) = \alpha(s) + (c-s)V_1(s), \tag{1.1}$$

where c is a real constant [12]. Throughout this article will be taken $c - s \neq 0$ for convenience.
If a rigid body moves along a unit speed curve α, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector ω which satisfies $V_i' = \omega \times V_i$ $(i = 1,2,3)$. The vector ω is called the *Darboux vector*. In terms of Frenet vectors, Darboux vector is given by $\omega = \tau V_1 + \kappa V_3$ [13]. Also, we have $\kappa = \|\omega\|\cos\theta,\ \tau = \|\omega\|\sin\theta,$ where θ is the angle between the Darboux vector ω of α and binormal vector V_3. Observe that $\theta = \arctan\frac{\tau}{\kappa}$.
Let $\{V_1(s), V_2(s), V_3(s)\}$ and $\{V_1^*(s), V_2^*(s), V_3^*(s)\}$ are Frenet frames of the curves α and β, respectively. If the curve β is the involute of α then we have

$$\begin{pmatrix} V_1^*(s) \\ V_2^*(s) \\ V_3^*(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{pmatrix}, [14]. \tag{1.2}$$

2. Surface family with a common involute geodesic

Suppose we are given a unit speed parametric curve $\alpha = \alpha(s)$ so that $\|\alpha''(s)\| \neq 0$, in 3-dimensional space. Let β be an involute of the given curve α. Surface family that possesses β as a common curve is given in the parametric form as

$$P(s,t) = \beta(s) + u(s,t)V_1^*(s) + v(s,t)V_2^*(s) + w(s,t)V_3^*(s), \tag{2.1}$$

where $u(s,t),\ v(s,t)$ and $w(s,t)$ are C^1 functions and are called *marching-scale functions* and $\{V_1^*(s), V_2^*(s), V_3^*(s)\}$ is the Frenet frame of the curve β. Using Eqn. (1.2) we can express Eqn. (2.1) in terms of Frenet frame $\{V_1(s), V_2(s), V_3(s)\}$ of the curve α as

$$P(s,t) = \beta(s) + (-v(s,t)\cos\theta + w(s,t)\sin\theta)V_1(s) + u(s,t)V_2(s) + (v(s,t)\sin\theta + w(s,t)\cos\theta)V_3(s), \tag{2.2}$$

where $n_1 \leq s \leq n_2,\ m_1 \leq t \leq m_2$.

Remark 2.1. *Observe that choosing different marching-scale functions yields different surfaces possessing β as a common curve.*

Our goal is to find the necessary and sufficient conditions for which the curve β is isoparametric and geodesic on the surface $P(s,t)$. Firstly, as β is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t = t_o \in [m_1, m_2]$ such that $P(s,t_o) = \beta(s)$, that is,

$$u(s,t_0) = v(s,t_0) = w(s,t_0) = 0. \tag{2.3}$$

Secondly the curve β is geodesic on the surface $P(s,t)$ if and only if along the curve the surface normal vector field $N(s,t_0)$ is parallel to the principal normal vector field V_2^* of the curve β. The normal vector of $P(s,t)$ can be written as

$$N(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

This equation can be expressed in terms of (1.2) and (2.2) as

$$N(s,t_0) = \kappa(c-s)\left[-\frac{\partial w}{\partial t}(s,t_0)V_2^* * (s) + \frac{\partial v}{\partial t}(s,t_0)V_3^*(s)\right],$$

where κ is the curvature of the curve α. Since $\kappa(s) \neq 0$, the curve β is a geodesic on the surface $P(s,t)$ if and only if

$$\frac{\partial w}{\partial t}(s,t_0) \neq 0,\ \frac{\partial v}{\partial t}(s,t_0) = 0.$$

So, we can present :

Theorem 2.2. *Let α be a unit speed curve with nonvanishing curvature and β be its involute. β is a geodesic on the surface $P(s,t)$if and only if*

$$\begin{cases} u(s,t_0) = v(s,t_0) = w(s,t_0) = 0, \\ \frac{\partial w}{\partial t}(s,t_0) \neq 0,\ \frac{\partial v}{\partial t}(s,t_0) = 0. \end{cases} \tag{2.4}$$

Corollary 2.3. *Let α be a unit speed curve with nonvanishing curvature and β be its involute. There exists a ruled surface possessing β as a geodesic.*

Proof. If we choose marching scale functions as

$$u(s,t) = v(s,t) \equiv 0,\ w(s,t) = t - t_0,$$

or

$$u(s,t) = w(s,t) = t - t_0,\ v(s,t) \equiv 0,$$

we obtain the ruled surfaces

$$P(s,t) = \beta(s) + (t - t_0) V_3^*(s), \tag{2.5}$$

or

$$P(s,t) = \beta(s) + (t - t_0)\left[V_1^*(s) + V_3^*(s)\right], \tag{2.6}$$

respectively. So, these ruled surfaces satisfy Eqn. (2.4) and β is a geodesic on them.

□

Corollary 2.4. *Ruled surface (2.5) is developable if and only if*

$$\theta(s) = s + c,$$

where c is a constant.

Corollary 2.5. *Ruled surface (2.6) is developable if and only if α is a helix.*

3. Examples

3.1. Example 1

Let us take the unit speed circle $\alpha(s) = (\cos s, \sin s, 0)$. Then, it is easy to show that

$$\begin{aligned} V_1(s) &= (-\sin s, \cos s, 0), \\ V_2(s) &= (-\cos s, -\sin s, 0), \\ V_3(s) &= (0,0,1), \\ \kappa &= 1,\ \tau = 0,\ \theta = 0. \end{aligned}$$

Letting $c = 0$ in Eqn. (1.1), we have

$$\beta(s) = (\cos s + s\sin s, \sin s - s\cos s, 0),$$

as an involute of α with Frenet vectors

$$\begin{aligned} V_1^*(s) &= (-\cos s, -\sin s, 0), \\ V_2^*(s) &= (\sin s, -\cos s, 0), \\ V_3^*(s) &= (0,0,1). \end{aligned}$$

If we choose $u(s,t) = v(s,t) \equiv 0,\ w(s,t) = t$, then according to Corollary 2.3 we get the ruled surface

$$\begin{aligned} P_1(s,t) &= \beta(s) + tV_3^*(s) \\ &= (\cos s + s\sin s, \sin s - s\cos s, t), \end{aligned}$$

$0 < s \leq 5,\ -5 \leq t \leq 5$, possessing β as a geodesic (*Fig.* 2).
For the same curve, if we choose $u(s,t) = w(s,t) = t,\ v(s,t) \equiv 0$ we obtain the ruled surface

$$\begin{aligned} P_2(s,t) &= \beta(s) + t\left[V_1^*(s) + V_3^*(s)\right] \\ &= ((1-t)\cos s + s\sin s, (1-t)\sin s - s\cos s, t), \end{aligned}$$

$0 < s \leq 5,\ -5 \leq t \leq 5$, satisfying Corollary 2.3 and possessing β as an involute geodesic (*Fig.* 3).

For the same curve, if we let $u(s,t) = e^{2t} - 1,\ v(s,t) \equiv 0,\ w(s,t) = t$, then Eqn. (2.4) is satisfied and we obtain

$$\begin{aligned} P_3(s,t) &= \beta(s) + \left(e^{2t} - 1\right) V_1^*(s) + tV_1^*(s) \\ &= \left(\left(2 - e^{2t}\right)\cos s + s\sin s, \left(2 - e^{2t}\right)\sin s - s\cos s, t\right), \end{aligned}$$

$0 < s \leq 5,\ -1 \leq t \leq 1$, as a member of the surface family possessing β as an involute geodesic (*Fig.* 4).

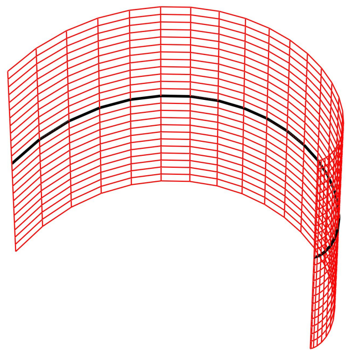

Fig. 2. Ruled surface $P_1(s,t)$ as a member of the surface family and its common involute geodesic β.

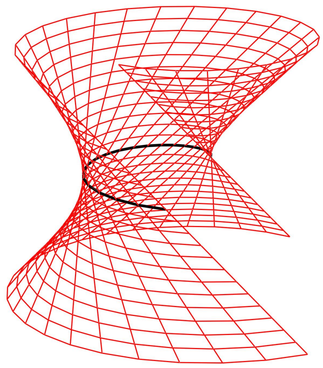

Fig. 3. Ruled surface $P_2(s,t)$ as a member of the surface family and its common involute geodesic β.

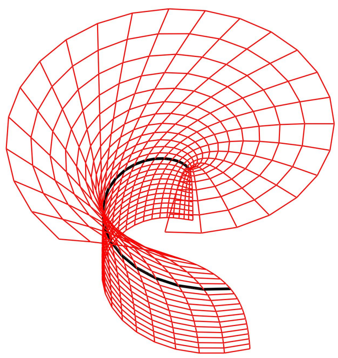

Fig. 4. $P_3(s,t)$ as a member of the surface family and its common involute geodesic β.

3.2. Example 2

Let $\alpha(s)=\left(a_1\cos\frac{s}{a_3},a_1\sin\frac{s}{a_3},\frac{a_2 s}{a_3}\right)$ be an arc length helix, where $a_1,a_2,a_3\in\mathbb{R},\ a_1^2+a_2^2=a_3^2,\ a_1>0$. One can show that

$$\begin{aligned}
V_1(s) &= \left(-\frac{a_1}{a_3}\sin\frac{s}{a_3},\frac{a_1}{a_3}\cos\frac{s}{a_3},\frac{a_2}{a_3}\right),\\
V_2(s) &= \left(-\cos\frac{s}{a_3},-\sin\frac{s}{a_3},0\right),\\
V_3(s) &= \left(\frac{a_2}{a_3}\sin\frac{s}{a_3},-\frac{a_2}{a_3}\cos\frac{s}{a_3},\frac{a_1}{a_3}\right),\\
\kappa &= \frac{a_1}{a_3^2},\ \tau=\frac{a_2}{a_3^2},\ \theta=\arctan\frac{a_2}{a_1}.
\end{aligned}$$

So we have

$$\beta(s) = \left(a_1\cos\frac{s}{a_3} - \frac{a_1}{a_3}(c-s)\sin\frac{s}{a_3},\; a_1\sin\frac{s}{a_3} + \frac{a_1}{a_3}(c-s)\cos\frac{s}{a_3}, \frac{ca_2}{a_3}\right)$$

as an involute of α with Frenet vectors

$$\begin{aligned} V_1^*(s) &= \left(-\cos\frac{s}{a_3}, -\sin\frac{s}{a_3}, 0\right), \\ V_2^*(s) &= sgn(a_3)\left(\sin\frac{s}{a_3}, -\cos\frac{s}{a_3}, 0\right), \\ V_3^*(s) &= (0,0,sgn(a_3)). \end{aligned}$$

Taking $a_1 = \frac{\sqrt{3}}{2}$, $a_2 = \frac{1}{2}$, $a_3 = 1$ results in $\theta = \frac{\pi}{6}$ and if we let $c = \sqrt{3}$ in formula (1.1) we get

$$\beta(s) = \left(\frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right)\sin s,\; \frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right)\cos s, \frac{\sqrt{3}}{2}\right).$$

If we let $u(s,t) \equiv 0$, $v(s,t) = \sqrt{3}t$, $w(s,t) = t$, then Eqn. (2.4) is satisfied and we have

$$P_4(s,t) = \left(\frac{\sqrt{3}}{2}\cos s - \left(\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) - \sqrt{3}t\right)\sin s,\; \frac{\sqrt{3}}{2}\sin s + \left(\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) - \sqrt{3}t\right)\cos s, \frac{\sqrt{3}}{2} + t\right),$$

$-1,6 \leq s \leq 1,6$, $-3 \leq t \leq 3$, as a member of surface family possessing β as an involute geodesic $(Fig.5)$.

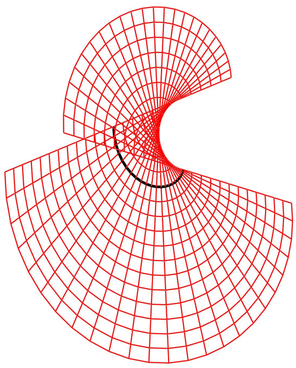

Fig. 5. $P_4(s,t)$ as a member of the surface family and its common involute geodesic β.

For the same curve if we let $u(s,t) = \tan t$, $v(s,t) = \sqrt{3}(e^t - 1)$, $w(s,t) = (e^t - 1)$, then Eqn. (**??**) is satisfied and we get

$$P_5(s,t) = \left(\left(\frac{\sqrt{3}}{2} - \tan t\right)\cos s - \left(\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) - \sqrt{3}(e^t-1)\right)\sin s,\; \left(\frac{\sqrt{3}}{2} - \tan t\right)\sin s + \left(\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) - \sqrt{3}(e^t-1)\right)\cos s, \frac{\sqrt{3}}{2} + e^t - 1\right),$$

$-1,6 \leq s \leq 1,6$, $-0.6 \leq t \leq 0.6$, as a member of the surface family accepting β as an involute geodesic $(Fig.\ 6)$.
If we choose $u(s,t) = s\tan t$, $v(s,t) = \sqrt{3}s\sin t$, $w(s,t) = s\sin t$, then Eqn. (2.4) is satisfied and we get

$$P_6(s,t) = \left(\left(\frac{\sqrt{3}}{2} - s\tan t\right)\cos s - \left(\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) - \sqrt{3}s\sin t\right)\sin s,\; \left(\frac{\sqrt{3}}{2} - s\tan t\right)\sin s + \left(\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) - \sqrt{3}s\sin t\right)\cos s, \frac{\sqrt{3}}{2} + s\sin t\right),$$

$0 < s \leq 1,6$, $-0,3 \leq t \leq 0,3$, as a member of the surface family accepting β as an involute geodesic $(Fig.\ 7)$.

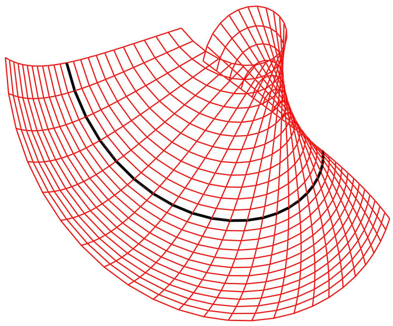

Fig. 6. $P_5(s,t)$ as a member of the surface family and its common involute geodesic β.

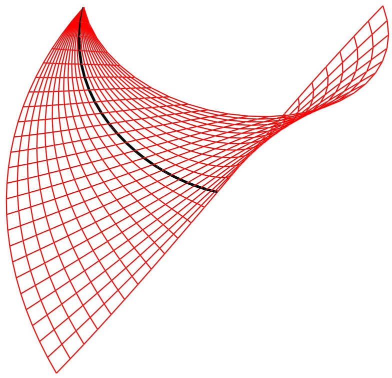

Fig. 7. $P_6(s,t)$ as a member of the surface family and its common involute geodesic β.

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