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Research Article

On Some Properties of Distance in TO-Space

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Abstract

The aim of this work is to investigate some properties of the truncated octahedron metric introduced in the space in further studies on metric geometry. With this metric, the 3-dimensional analytical space is a Minkowski geometry which is a non-Euclidean geometry in a finite number of dimensions. In a Minkowski geometry, the unit ball is a certain symmetric closed convex set instead of the usual sphere in Euclidean space. The unit ball of the truncated octahedron geometry is a truncated octahedron which is an Archimedean solid. In this study, first, metric properties of truncated octahedron distance, d_{TO}, in $\mathbb{R}^2$ has been examined by metric approach. Then, by using synthetic approach some distance formulae in $\mathbb{R}^3_{TO}$, 3-dimensional analytical space furnished with the truncated octahedron metric has been found.

Keywords

Metric, Convex polyhedra, Truncated octahedron, Distance of a point to a line, Distance of a point to a plane, Distance between two lines

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1. INTRODUCTION

The planar shape restricted with the line segments in the finite number is called a "polygon". A polyhedron is a three-dimensional figure consists of polygons. To define polyhedra the terms faces, edges and vertices are used. Polygonal parts of a polyhedron are called its faces. A line segment is called an edge, along which two faces come together. A point is called a vertex where several edges and faces come together. So, a polyhedron is a three dimensional solid with flat faces, straight edges and vertices.

Polyhedra have been studied by mathematicians and geometers during many years, because of their symmetries. There are many philosophers that worked on polyhedra among the ancient Greeks. The reason of Polyhedra attract people's interest is that polyhedral shapes are widely found in the nature. The kernels of some nuts and fruits contain many small seeds which grow in a restricted space. Pomegranates are one example. As each seed grows it presses up against its neighbours. The seeds prevent each other from expanding uniformly and they grow to fill the available space producing flat-faced seeds with sharp corners. If the seeds had a perfectly uniform distribution before they began to grow and were subjected to isotropic compression forces they would end up as rhombic dodecahedra. The principal of economy –maximising volume from given materials – leads to the construction of roughly spherical organisms. These sometimes have polyhedral substructures. Ernst Haeckel on his voyage on H.M.S. Challenger, in the 1880's, drew many pictures of microscopic single-celled creatures called radiolaria. A radiolarian has a spherical skeleton that is polyhedral in character. Haeckel named three of them circoporus octahedrus, circorrhegma dodecahedra and circogonia icosahedra because he thougt they resembled the Platonic solids. The recently discovered allotrope of carbon also forms polyhedral spheres, ellipsoids and tubes. In the smallest example, C60 , the sixty atoms are arranged in the same pattern as the vertices of a truncated icosahedron-familiar as a soccer ball. Polyhedral molecules have been known for some time. Organic chemists have made carbon-hydrogen structures such as cubane, C8H8, whose carbon atoms lie at the corners of a cube [1].

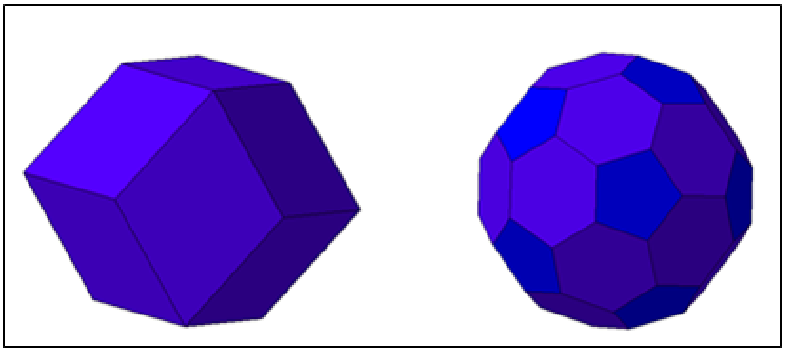

Figure 1. (a) Rhombic dodecahedron (b) truncated icosahedron

When we refer to polyhedron as a whole, that is, when we talk about a point or a polygon in a polyhedron, we can call this polyhedron a solid. Also, if the whole of the line segment connecting any two points remains on or in the surface of the polyhedron, this polyhedron is called convex, otherwise it is called concave. Geometrically, convexity of a polyhedron can be defined as a line connecting any two points of the polyhedron always lays in the interior of the polyhedron or on the surface of it.

A polyhedron with congruent faces and identical vertices is called a regular polyhedron. Regular and convex solids are called Platonic solids and there are only five. They are called Platonic solids because they are firstly described by Plato in his “Timaeus”. Semi-regular convex polyhedra which faces consist of two or more different types of regular polygons meeting in identical vertices are called Archimedean solids. And there are thirteen of them. Catalan solids are exactly thirteen just like Archimedean solids because they are dual polyhedra of the Archimedean solids and all are convex. They named after contruction of the dual solids of the Archimedean solids was completed in 1865 by Catalan. Faces of the Catalan solids are not regular polygons unlike Platonic and Archimedean solids.

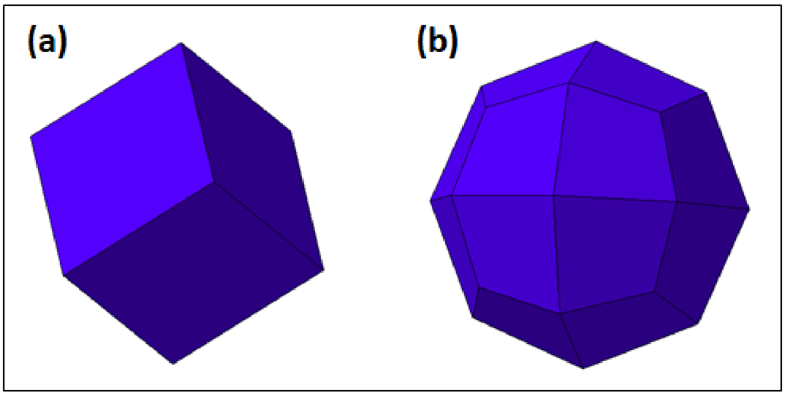

Figure 2. (a) Cube (b) deltoidal icositetrahedron

Polyhedra, especially convex ones, have been studied by geometers for thousands of years because of their symmetries. Also metric space geometry is studied and improved by some mathematicians. In these studies it had been found that spheres of some metrics are certain convex solids. In taxicab space the unit sphere is an octahedron which is a Platonic solid; in maximum space the unit sphere is a cube which is another Platonic solid, and in CC-space the unit sphere is a deltoidal icositetrahedron which is a Catalan solid. Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions. Here the linear structure is same as the Euclidean one, but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [2]. The mentioned space geometries are examples of Minkowski geometries. It is easy to find unit sphere of a geometry when the metric is known. In Refs. [3-9], the authors have given some metrics which spheres are some of Platonic, Archimedean and Catalan solids by a reverse question; “If the sphere is known, then what is the metric of this geometry?”. So there are some metrics which unit spheres are convex polyhedra.

In Ref [4], the truncated octahedron metric is introduced for 3-dimensional analytical space. By projection of truncated octahedron metric for 3-dimensional analytical space to 2-dimensional analytical plane, truncated octahedron distance, d_{TO} can be defined as

$$d_{TO}(P_1, P_2) = max\left\{|x_1 - x_2|, |y_1 - y_2|, \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2|)\right\}$$

for the points $P_1, P_2 \in \mathbb{R}^2$. If $\mathcal{L}_E$ is the set of all lines in the Cartesian coordinate plane, and m_E is the standard angle measure function in the Euclidean plane, then $\{\mathbb{R}^2, \mathcal{L}_E, d_{TO}, m_E\}$ called TO-plane, is a model of protractor geometry. (This can be shown easily: the proof is similar to that of taxicab plane; refer to [10] or [11] to see that the taxicab plane is a model of protractor geometry.) TO-plane is also in the class of non-Euclidean geometries since it fails to satisfy the side-angle-side axiom. However, TO-plane is almost the same as Euclidean plane $\{\mathbb{R}^2, \mathcal{L}_E, d_E, m_E\}$ since the points are the same, the lines are the same and the angles are measured in the same way. Since the TO-plane ($\mathbb{R}^2_{TO}$) geometry has a different distance function it seems interesting to study the TO-analogues of the topics that include the concepts of distance in the Euclidean geometry.

By these motivations, in this study, first it is shown that TO-plane geometry consisting of $\mathcal{P}_E = \mathbb{R}^2$, $\mathcal{L}_E$ and d_{TO} is a metric geometry. Then distance of a point to a line in the plane is found. Also in truncated octahedron space some other distance formulae are found such as distance of a point to a line, distance of a point to a plane and distance between two lines by a similiar process used in ref. [12] and which is different from refs. [13] and [14].

2. TO-Plane Geometry

The truncated octahedron metric is introduced in ref. [4] for 3-dimensional analytical space and for the plane this distance, d_{TO} can be defined as

$$d_{TO}(P_1, P_2) = max\left\{|x_1 - x_2|, |y_1 - y_2|, \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2|)\right\}$$

where $P_1, P_2 \in \mathbb{R}^2$.

Theorem 2.1. The distance function d_{TO} is a metric in $\mathbb{R}^2$.

Proof. Let $d_{TO}: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $P_1 = (x_{1,}y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$ be any three points in $\mathbb{R}^2$. To prove that d_{TO} is a metric in $\mathbb{R}^2$, the following axioms must be provided by all P_1, P_2 and $P_3 \in \mathbb{R}^2$;

M1) $d_{TO}(P_1, P_2) \geq 0$ ve $d_{TO}(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$

M2) $d_{TO}(P_1, P_2) = d_{TO}(P_2, P_1)$

M3) $d_{TO}(P_1, P_3) \leq d_{TO}(P_1, P_2) + d_{TO}(P_1, P_3)$.

M1) By the definition of the absolute value $d_{TO}(P_1, P_2) \geq 0$. If $d_{TO}(P_1, P_2) = 0$, then according to truncated octahedron distance function three cases are possible. For example if $d_{TO}(P_1, P_2) = |x_1 - x_2|$, then

$$\begin{aligned} d_{TO}(P_1, P_2) = 0 \quad &\Leftrightarrow \quad |x_1 - x_2| = 0, |y_1 - y_2| = 0 \\ &\Leftrightarrow \quad x_1 = x_2 \, , y_1 = y_2 \\ &\Leftrightarrow \quad P_1 = P_2 \end{aligned}$$

The other cases can be easily shown by similar way. Thus $d_{TO}(P_1, P_2) = 0$ iff $P_1 = P_2$ is obtained.

M2) By the definition of absolute value $|x_i - x_j| = |x_j - x_i|$, $|y_i - y_j| = |y_j - y_i|$ for all $x_i, x_j \in \mathbb{R}^2$ and $i, j \in \{1,2\}$. Therefore one can get $d_{TO}(P_1, P_2) = d_{TO}(P_{2,}P_1)$.

M3) To show that $d_{TO}(P_1, P_2) \leq d_{TO}(P_1, P_3) + d_{TO}(P_3, P_2)$ twenty seven subcases must be considered. Let $d_{TO}(P_1, P_2) = |x_1 - y_1|$, $d_{TO}(P_1, P_3) = |x_1 - z_1|$ and $d_{TO}(P_3, P_2) = \frac{2}{3}(|z_1 - y_1| + |z_2 - y_2|)$. Thus

$$\begin{aligned} d_{TO}(P_1, P_2) \quad &= \quad |x_2 - y_2| = |x_2 - x_3 + x_3 - y_2| \\ &\leq \quad |x_1 - x_3| + |x_3 - x_2| \\ &\leq \quad |x_1 - x_3| + \frac{2}{3}(|x_3 - x_2| + |y_3 - y_2|) \end{aligned}$$

since $|x_3 - x_2| \leq \frac{2}{3}(|x_3 - x_2| + |y_3 - y_2|)$. So $d_{TO}(P_1, P_2) \leq d_{TO}(P_1, P_3) + d_{TO}(P_3, P_2)$ is obtained. Other subcases would be shown by similarly.

Theorem 2.2. Cartesian plane with distance function

$$d_{TO}(P, Q) = max\left\{|x_1 - x_2|, |y_1 - y_2|, \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2|)\right\}$$

is a metric geometry, where $P = (x_1, y_1)$, $Q = (x_2, y_2) \in \mathbb{R}^2$.

Proof. Let ℓ be a line in $\mathbb{R}^2_{TO}$. Then a ruler for ℓ must be found. Lines of $\mathbb{R}^2_{TO}$ are of the form $L_a = \{(x,y) \in \mathbb{R}^2_{TO} : x = a\}$ and $L_{m,b} = \{(x,y) \in \mathbb{R}^2_{TO} : y = mx + b\}$. Then two cases are possible;

Case I : Let ℓ be a vertical line as $\ell = L_a$. For $P \in L_a$, $f_{TO}(P) = f_{TO}(a,y) = y$ be defined. It must be shown that f_{TO} is one to one, surjective and holds the ruler axiom. For $\mathrm{P}, \mathrm{Q} \in \ell$ it must be hold that if $f_{TO}(P) = f_{TO}(Q)$ then $P = Q$. Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$. Since $x_1 = x_2 = a$, then $P = (a, y_1)$ and $Q = (a, y_2)$ can be written. If $f_{TO}(P) = f_{TO}(Q)$, then $y_1 = y_2$ and $P = Q$ holds. And f_{TO} is surjective since $f_{TO}(a,t) = t$ and $(a,t) \in \ell$ for $\forall t \in \mathbb{R}$. For ruler axiom, it must be shown that $|f_{TO}(P) - f_{TO}(Q)| = d_{TO}(P,Q)$. Since $f_{TO}(P) = y_1$ and $f_{TO}(Q) = y_2$ then $|f_{TO}(P) - f_{TO}(Q)| = |y_1 - y_2|$ and $d_{TO}(P,Q) = |y_1 - y_2|$. Thus $d_{TO}(P,Q) = |f_{TO}(P) - f_{TO}(Q)|$ and ruler axiom holds.

Case II : Let $\ell = L_{m,b}$ and f_{TO} function be defined as

$$f_{TO}(x,y) = \begin{cases} x & , \quad |m| \leq \dfrac{1}{2} \\ \dfrac{2}{3}(1+|m|)x & , \quad \dfrac{1}{2} \leq |m| < 2 \\ |m|x & , \quad |m| \geq 2 \end{cases}$$

It must be shown that f_{TO} is one to one, surjective and holds the ruler axiom. So three cases of m must be considered. For example let $\frac{1}{2} \leq |m| < 2$. $\ell = L_{m,b}$ means that $y = mx + b$. For $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ it is $f_{TO}(P) = \frac{2}{3}(1+|m|)x_1$ and $f_{TO}(Q) = \frac{2}{3}(1+|m|)x_2$. If $f_{TO}(P) = f_{TO}(Q)$, then $x_1 = x_2$. Since $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, $y_1 = y_2$ is obtained. Thus $P = Q$ holds. Since when $f_{TO}\left(\frac{3t}{2(1+|m|)}, y\right)$ for every $t \in \mathbb{R}$ there exists a $x \in \mathbb{R}$ which satisfies $x = \frac{2}{3}(1+|m|)\frac{3}{2}\frac{t}{(1+|m|)} = t$, f_{TO} is surjective. For ruler axiom $|f_{TO}(P) - f_{TO}(Q)| = d_{TO}(P,Q)$ must be hold. Since $f_{TO}(P) = \frac{2}{3}(1+|m|)x_1$ and $f_{TO}(Q) = \frac{2}{3}(1+|m|)x_2$ then $|f_{TO}(P) - f_{TO}(Q)| = \frac{2}{3}(1+|m|)|x_1 - x_2|$ and

$$\begin{aligned} d_{TO}(P,Q) &= max\left\{ \begin{matrix} |x_1 - x_2|, |y_1 - y_2|, \\ \dfrac{2}{3}(|x_1 - x_2| + |y_1 - y_2|) \end{matrix} \right\} \\ &= max\left\{ \begin{matrix} |x_1 - x_2|, |m||x_1 - x_2|, \\ \dfrac{2}{3}(|x_1 - x_2| + |m||x_1 - x_2|) \end{matrix} \right\} \\ &= \frac{2}{3}(1+|m|)|x_1 - x_2| \end{aligned}$$

is obtained. Thus $|f_{TO}(P)-f_{TO}(Q)|=d_{TO}(P,Q)$ and ruler axiom holds.

Truncated octahedron circle which center is $C=(x_0,y_0)$ and radius is r in the TO- plane can be defined as

$$\mathcal{C}_{TO}=\left\{(x,y)\,|\;max\left\{|x-x_0|,|y-y_0|,\frac{2}{3}(|x-x_0|+|y-y_0|)\right\}=r\right\}$$

Coordinates of vertex of a truncated octahedron circle with center $C=(x_0,y_0)$ are $V_1=(x_0-\frac{r}{2}\,,y_0+r)$, $V_2=\left(x_0-r,y_0+\frac{r}{2}\right)$, $V_3=\left(x_0-r,y_0-\frac{r}{2}\right)$, $V_4=\left(x_0-\frac{r}{2},y_0-r\right)$, $V_5=\left(x_0+\frac{r}{2},y_0-r\right)$, $V_6=\left(x_0+r,y_0-\frac{r}{2}\right)$, $V_7=\left(x_0+r,y_0+\frac{r}{2}\right)$, $V_8=\left(x_0+\frac{r}{2},y_0+r\right)$.

Theorem 2.3. In TO-plane the distance of a point $P=(x_0,y_0)$ to a line $\ell : ax+by+c=0$ is

$$d_{TO}(P,\ell)=\frac{2|ax_0+by_0+c|}{max\{|a+2b|,|a-2b|,|2a+b|,|2a-b|\}}$$

Proof. Let $P=(x_0,y_0)$ is a point in the plane. To find TO-distance of the point to a line $\ell :$ $ax+by+c=0$ TO-circle is considered. While enlarging the radius of the P centered circle, intersection point of the circle and the line is being investigated, because distance between P and the intersection point of the line and circle gives the TO-distance of the point P to the line ℓ.

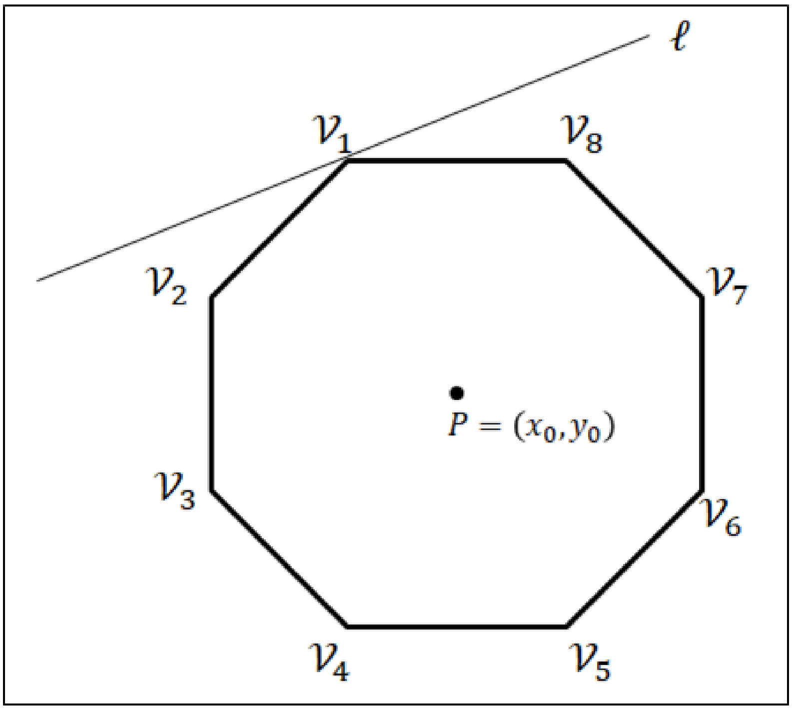

Figure 3. Intersection of TO-circle and the line ℓ

For example if the intersection point of the line and the circle is the vertex $V_1 = (x_0 - \frac{r}{2}, y_0 + r)$, then slope of the line PV_1 is -2. Thus the equation of the line PV_1 is $y = -2x + 2x_0 + y_0$. Intersection of the line $\ell: ax + by + c = 0$ and the line PV_1 must be found. By solving these equations together $(x, y) = \left(\frac{-2bx_0 - by_0 - c}{a-2b}, \frac{2ax_0 + ay_0 + 2c}{a-2b}\right)$ is obtained. So distance of the point (x,y) to the point (x_0, y_0) ;

$$\begin{aligned}
d_{TO}(P,\ell) &= d_{TO}\big((x,y),(x_0,y_0)\big) \\
&= \max\left\{|x-x_0|, |\mathrm{y}-y_0|, \frac{2}{3}(|x-x_0|+|\mathrm{y}-y_0|)\right\} \\
&= \max\left\{\begin{array}{l} \left|\dfrac{-2bx_0-by_0-c}{a-2b}-x_0\right|, \left|\dfrac{2ax_0+ay_0+2c}{a-2b}-y_0\right|, \\ \dfrac{2}{3}\left(\left|\dfrac{-2bx_0-by_0-c}{a-2b}-x_0\right|+\left|\dfrac{2ax_0+ay_0+2c}{a-2b}-y_0\right|\right) \end{array}\right\} \\
&= 2\left|\frac{ax_0+by_0+c}{a-2b}\right|
\end{aligned}$$

For other cases by similiar calculations, desired formula would be obtained.

3. TO-Space Geometry

Truncated octahedron is an Archimedean solid which has 14 faces, 36 edges and 24 vertices. 8 of its faces are regular hexagons and 6 of them are squares. This solid is obtained by truncating the octahedron.

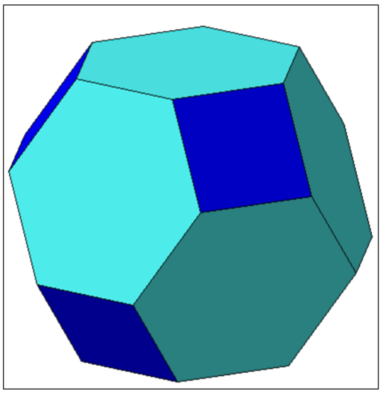

Figure 4. Truncated octahedron

Distance function of which unit sphere is truncated octahedron is denoted by d_{TO}. Now we give the definition of truncated octahedron distance and the theorem without proof which implies this distance function holds the metric axioms as in [4].

Definition 3.1. $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct points in $\mathbb{R}^3$. Distance function $d_{TO}: \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$defined by

$$d_{TO}(P_1,P_2) = max\left\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \frac{2}{3}\left(\left||x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right|\right)\right\}$$

is called the truncated octahedron distance.

Theorem 3.1. d_{TO} truncated octahedron distance function which is defined by

$$d_{TO}(P_1,P_2) = max\left\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \frac{2}{3}\left(\left||x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right|\right)\right\}$$

where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is a metric in 3-dimensional analitical space and unit sphere of this metric in $\mathbb{R}^3$ is a truncated octahedron.

In the analytical 3-space furnished by TO-metric the set of all points at 1 TO-distance from the origin is

$$S_{TO} = \left\{(x, y, z): d_{TO}(X, O) = max\left\{|x|, |y|, |z|, \frac{2}{3}(|x| + |y| + |z|)\right\} = 1\right\}$$

and locus of these points is a truncated octahedron.

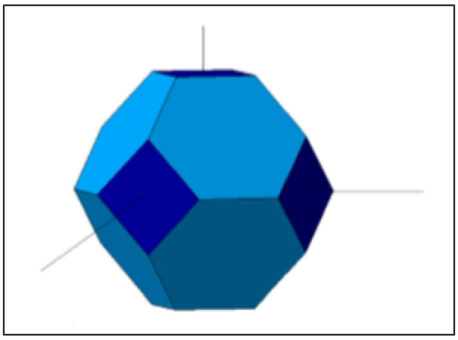

Figure 5. Unit sphere of TO-metric

Vertices of the unit sphere of TO-metric are $V_1 = \left(\frac{1}{2}, 0, 1\right)$, $V_2 = \left(\frac{1}{2}, 0, -1\right)$, $V_3 = \left(-\frac{1}{2}, 0, 1\right)$, $V_4 = \left(-\frac{1}{2}, 0, -1\right)$, $V_5 = \left(1, \frac{1}{2}, 0\right)$, $V_6 = \left(1, -\frac{1}{2}, 0\right)$, $V_7 = \left(-1, \frac{1}{2}, 0\right)$, $V_8 = \left(-1, -\frac{1}{2}, 0\right)$, $V_9 = \left(0, 1, \frac{1}{2}\right)$, $V_{10} = \left(0, 1, -\frac{1}{2}\right)$, $V_{11} = \left(0, -1, \frac{1}{2}\right)$, $V_{12} = \left(0, -1, -\frac{1}{2}\right)$, $V_{13} = \left(0, \frac{1}{2}, 1\right)$, $V_{14} = \left(0, \frac{1}{2}, -1\right)$, $V_{15} = \left(0, -\frac{1}{2}, 1\right)$, $V_{16} = \left(0, -\frac{1}{2}, -1\right)$, $V_{17} = \left(1, 0, \frac{1}{2}\right)$, $V_{18} = \left(1, 0, -\frac{1}{2}\right)$, $V_{19} = \left(-1, 0, \frac{1}{2}\right)$, $V_{20} = \left(-1, 0, -\frac{1}{2}\right)$, $V_{21} = \left(\frac{1}{2}, 1, 0\right)$, $V_{22} = \left(\frac{1}{2}, -1, 0\right)$, $V_{23} = \left(-\frac{1}{2}, 1, 0\right)$, $V_{24} = \left(-\frac{1}{2}, -1, 0\right)$

Theorem 3.2. Truncated Octahedron distance of a point $P = (x_0, y_0, z_0)$ to a plane $\mathcal{P}: Ax + By + Cz + D = 0$ is

$$d_{TO}(P, \mathcal{P}) = 2\frac{|Ax_0 + By_0 + Cz_0 + D|}{\alpha}$$

where

$$\alpha = max\begin{Bmatrix} |A+2B|, |A-2B|, |A+2C|, |A-2C|, |B+2A|, |B-2A|, \\ |B+2C|, |B-2C|, |C+2A|, |C-2A|, |C+2B|, |C-2B| \end{Bmatrix}$$

Proof. To find the truncated octahedron distance of the point $\mathrm{P}=(\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0)$ to the plane $\mathcal{P}: Ax+By+Cz+D=0$, a truncated octahedron with center P is considered. The intersection point of the truncated octahedron and the plane is being searched. While inflating the truncated octahedron, the truncated octahedron and the plane $\mathcal{P}$ would intersect at a corner, at an edge or at a face of the truncated octahedron. In fact all these possible situations would be reduced to intersection at a corner. So the direction vector of the line passing through P and one of the corner points of the truncated octahedron is an element of

Δ_1 ={(1,0,2), (1,0,-2), (-1,0,2), (-1,0,-2), (2,1,0), (2,-1,0), (-2,1,0), (-2,-1,0), (0,2,1), (0,2,-1), (0,-2,1), (0,-2,-1), (0,1,2), (0,1,-2), (0,-1,2), (0,-1,-2), (2,0,1), (2,0,-1), (-2,0,1), (-2,0,-1), (1,2,0), (1,-2,0), (-1,2,0), (-1,-2,0)}

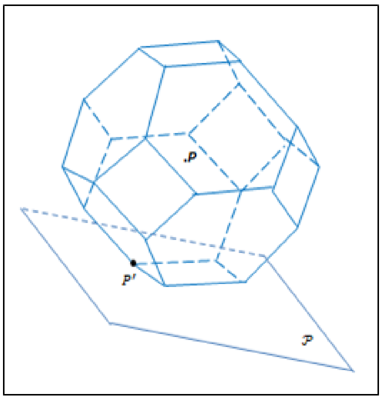

Figure 6. Intersection of a plane and the P-centered sphere in TO-space

For example if the direction of the line passing through $\mathrm{P}=(x_0, \mathrm{y}_0, z_0)$ is $(1,0,2)$ and the radius of the truncated octahedron is t_1, then

$$\frac{\mathrm{x}-\mathrm{x}_0}{1}=\frac{\mathrm{z}-\mathrm{z}_0}{2}=\mathrm{t}_1 \text{ and } \mathrm{y}=\mathrm{y}_0$$

is the standard equation of the line. And so $\mathrm{x}=\mathrm{x}_0+\mathrm{t}_1$, $\mathrm{y}=\mathrm{y}_0$ and $\mathrm{z}=\mathrm{z}_0+2\mathrm{t}_1$ for any point $P'=(x, \mathrm{y}, z)$ on the line. Since P' is both on the sphere and the plane

$$\mathrm{A}(\mathrm{x}_0+\mathrm{t}_1)+\mathrm{By}_0+\mathrm{C}(\mathrm{z}_0+2\mathrm{t}_1)+\mathrm{D}=0 \Rightarrow t_1=-\frac{(Ax_0+By_0+Cz_0+D)}{A+2C}$$

and $P'=(x,y,z)=(x_0+t_1, y_0, z_0+2t_1)$. Thus TO-distance between the point and the plane in this case is $\mathrm{d}_{\mathrm{TO}}(\mathrm{P}, \mathcal{P})=d_{TO}(P, P')=2|t_1|=2\left|\frac{Ax_0+By_0+Cz_0+D}{A+2C}\right|$

Similarly it can easily be computed for other directions in the set Δ and required formula is obtained.

Theorem 3.3. TO-distance of a point $P=(x_0,y_0,z_0)$ to a line ℓ given by $\frac{x-a}{p}=\frac{y-b}{q}=\frac{z-c}{r}$ is

$$d_{TO}(P,\ell)=\frac{max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}}{\Delta_2}$$

where

$$\begin{aligned}
\alpha_1 &= 2[(q-r)(x_0-a)-p(y_0-b)+p(z_0-c)]\\
\alpha_2 &= 2[(q+r)(x_0-a)-p(y_0-b)-p(z_0-c)]\\
\alpha_3 &= 2[q(x_0-a)-(p+r)(y_0-b)+q(z_0-c)]\\
\alpha_4 &= 2[q(x_0-a)-(p-r)(y_0-b)-q(z_0-c)]\\
\alpha_5 &= 2[r(x_0-a)+r(y_0-b)-(p+q)(z_0-c)]\\
\alpha_6 &= 2[r(x_0-a)-r(y_0-b)-(p-q)(z_0-c)]
\end{aligned}$$

and

$$\Delta_2=\begin{cases}
max\{|3p|,|p+2q-2r|,|p-2q+2r|\}, if\ \alpha_1>\alpha_i\ where\ i=2,3,4,5,6\\
max\{|3p|,|p+2q+2r|,|p-2q-2r|\}, if\ \alpha_2>\alpha_i\ where\ i=1,3,4,5,6\\
max\{|3q|,|2p+q+2r|,|2p-q+2r|\}, if\ \alpha_3>\alpha_i\ where\ i=1,2,4,5,6\\
max\{|3q|,|2p+q-2r|,|2p-q-2r|\}, if\ \alpha_4>\alpha_i\ where\ i=1,2,3,5,6\\
max\{|3r|,|2p+2q+r|,|2p+2q-r|\}, if\ \alpha_5>\alpha_i\ where\ i=1,2,3,4,6\\
max\{|3r|,|2p-2q+r|,|2p-2q-r|\}, if\ \alpha_6>\alpha_i\ where\ i=1,2,3,4,5
\end{cases}$$

Proof. To find the TO-distance of a point $\mathrm{P}=(\mathrm{x}_0,\mathrm{y}_0,\mathrm{z}_0)$ to a line $\ell\colon \frac{\mathrm{x-a}}{\mathrm{p}}=\frac{\mathrm{y-b}}{\mathrm{q}}=\frac{\mathrm{z-c}}{\mathrm{r}}=\mu$ where $\mu\in\mathbb{R}$, a truncated octahedron with center P is considered. The intersection point of the truncated octahedron and the line is being searched, because TO-distance between this intersection point and P gives the distance required. While inflating the truncated octahedron, the truncated octahedron and the line ℓ would intersect at a corner , at an edge or at a face of the truncated octahedron. In fact all these possible situations can be reduced to intersection at an edge.

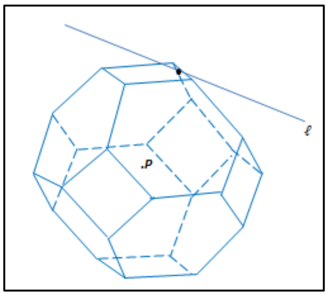

Figure 7. Intersection of a line and the P-centered sphere in TO-space

So 36 edges must be considered. Each of all these edges are on a line which is passing through

$\mathrm{P}_1=\left(\mathrm{x}_0+\mathrm{k},\mathrm{y}_0,\mathrm{z}_0+\frac{\mathrm{k}}{2}\right)$, $\mathrm{P}_2=\left(\mathrm{x}_0+\frac{\mathrm{k}}{2},\mathrm{y}_0+\mathrm{k},\mathrm{z}_0\right)$, $\mathrm{P}_3=\left(\mathrm{x}_0,\mathrm{y}_0+\frac{\mathrm{k}}{2},\mathrm{z}_0+\mathrm{k}\right)$,

$P_4 = \left(x_0, y_0 - \frac{k}{2}, z_0 + k\right)$, $P_5 = \left(x_0 + \frac{k}{2}, y_0 - k, z_0\right)$, $P_6 = \left(x_0 - \frac{k}{2}, y_0 - k, z_0\right)$, $P_7 = \left(x_0, y_0 - \frac{k}{2}, z_0 - k\right)$, $P_8 = \left(x_0 + k, y_0, z_0 - \frac{k}{2}\right)$, $P_9 = \left(x_0, y_0 + \frac{k}{2}, z_0 - k\right)$ $P_{10} = \left(x_0 - k, y_0, z_0 - \frac{k}{2}\right)$, $P_{11} = \left(x_0 - \frac{k}{2}, y_0 + k, z_0\right)$ or $P_{12} = \left(x_0 - k, y_0, z_0 + \frac{k}{2}\right)$ and has a direction vector $(1,0,-1)$, $(0,-1,1)$, $(0,1,1)$, $(-1,1,0)$, $(1,0,1)$ or $(1,1,0)$. For example direction vectors of the edges of the truncated octahedron sphere passing through the vertice $P_1 = (x_0 + k, y_0, z_0 + \frac{k}{2})$ would be $(1,0,-1)$, $(0,-1,1)$ or $(0,1,1)$. The edge with direction vector $(1,0,-1)$ can be expressed as

$$\frac{x - (x_0 + k)}{1} = \frac{z - \left(z_0 + \frac{k}{2}\right)}{-1} = \lambda\,, y = y_0$$

where $\lambda \in \mathbb{R}$. Coordinates of any point on this edge are in the form $x = x_0 + k + \lambda$, $y = y_0$ and $z = z_0 + \frac{k}{2} - \lambda$. Also the coordinates of any point on the line ℓ can be given in the form $x = p\mu + a$, $y = q\mu + b$ and $z = r\mu + c$. Now the value of k, which is the radius of truncated octahedron sphere at the moment of the intersection of the line and the edge is being searched. Since x, y and z are coordinates of the intersection point

$$x_0 + k + \lambda = p\mu + a$$
$$y_0 = q\mu + b$$
$$z_0 + \frac{k}{2} - \lambda = r\mu + c$$

equations hold. By this equation system the value of k is found as

$$k = -\frac{2}{3}\left[\frac{q(x_0 - a) - (p + r)(y_0 - b) + q(z_0 - c)}{q}\right]$$

For other choices of P_i , i=2,3,..,12 , k would be found by the similiar way and required formula is obtained.

Theorem 3.4. Truncated octahedron distance between any two skew lines given by $\ell: \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} = \lambda$ and $\ell': \frac{x-a'}{p'} = \frac{y-b'}{q'} = \frac{z-c'}{r'} = \mu$ is

$$d_{TO}(\ell, \ell') = \frac{2|\gamma(a - a') + \beta(b - b') + \alpha(c - c')|}{\max\left\{\begin{matrix} |2\alpha + \beta|, |2\alpha - \beta|, |2\alpha + \gamma|, |2\alpha - \gamma|, |2\beta + \alpha|, |2\beta - \alpha|, \\ |2\beta + \gamma|, |2\beta - \gamma|, |2\gamma + \alpha|, |2\gamma - \alpha|, |2\gamma + \beta|, |2\gamma - \beta| \end{matrix}\right\}}$$

where α=(p'q-pq'), β=(pr'-p'r), γ=(rq'-r'q).

Proof: Let $\ell: \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} = \lambda$ and $\ell': \frac{x-a'}{p'} = \frac{y-b'}{q'} = \frac{z-c'}{r'} = \mu$ be two lines. The distance between two lines can be expressed as $d_{TO}(\ell, \ell') = \min\{ d_{TO}(P, P')\}$ where P is a point on the line ℓ and P' is a point on the line ℓ'. If these lines are parallel then TO-distance between these lines can be found easily by the distance of a point to a line as $d_{TO}(\ell, \ell') = d_{TO}(P, \ell')$ where $P = (a, b, c)$ is a point on the line ℓ. Let ℓ and ℓ' be skew lines. Then at least one of $pq' - p'q$, $qr' - q'r$ and $p'r - pr'$ is not zero. Otherwise these lines would be parallel to each other. Consider the points P and P' which are on the line ℓ and ℓ', respectively. So $P = (p\lambda + a, q\lambda + b, r\lambda + c)$ and $P' = (p'\mu + a', q'\mu + b', r'\mu + c')$. To obtain $\min\{d_{TO}(P, P')\}$, the direction of the line passing through points P and P' must be an element of

Δ_3={(0,1,2), (0,2,1), (0,2,-1), (0,1,-2), (1,2,0), (1,-2,0), (1,0,2), (1,0,-2), (2,1,0), (2,-1,0), (2,0,1), (2,0,-1)} and elements of Δ_3 are directions of diagonals of the truncated octahedron.

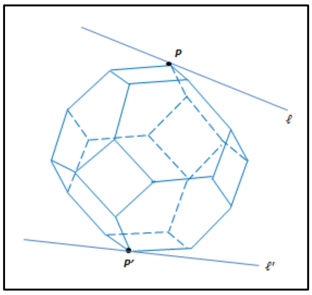

Figure 8. A TO-sphere which intersects both ℓ and ℓ'

For example the direction vector of the line passing through the points P and P' be (0,1,2) and the vector PP' is of the form $PP' = (p\lambda - p'\mu + a - a', q\lambda - q'\mu + b - b', r\lambda - r'\mu + c - c')$. Thus

$$\frac{q\lambda - q'\mu + b - b'}{1} = \frac{r\lambda - r'\mu + c - c'}{2}, p\lambda - p'\mu + a - a' = 0$$

By this equation system

$$\lambda = \frac{(2q' - r')(a - a') - 2p'(b - b') + p'(c - c')}{p'(2q - r) - p(2q' - r')}$$

and

$$\mu = \frac{(2q - r)(a - a') - 2p(b - b') + p(c - c')}{p'(2q - r) - p(2q' - r')}$$

are obtained. If values of λ and μ are used in

$$d_{TO}(P, P') = \max\left\{\begin{matrix} |p\lambda - p'\mu + a - a'|, |q\lambda - q'\mu + b - b|, |r\lambda - r'\mu + c - c'| \\ \frac{2}{3}(|p\lambda - p'\mu + a - a'| + |q\lambda - q'\mu + b - b| + |r\lambda - r'\mu + c - c'|) \end{matrix}\right\}$$

then

$$d_{TO}(P,P') = 2\left|\frac{(q'r - qr')(a - a') + (pr' - p'r)(b - b') + (p'q - pq')(c - c')}{p'(2q - r) - p(2q' - r')}\right|$$

is found. By similiar calculations for the rest of the directions, the required formula would be obtained.

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