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Research Article

Convergence and Stability Analysis of a New Four-Step Fixed-Point Algorithm

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Abstract

The concept of stability is studied on many different types of mathematical structures. This concept can be thought of as the small changes that will be applied in the structure studied should not disrupt the functioning of this structure. In this context, we performed the convergence and stability analysis of the new four-step iteration algorithm that we defined in this study, under appropriate conditions. In addition, we execute a speed comparison with existing algorithms to prove that the new algorithm is effective and useful, and we gave a numerical example to support our result.

Keywords

Stability, Convergence, Fixed-point algorithm, Convergence speed

1. INTRODUCTION

Fixed point theory is a field with wide applications, which is located at the intersection of many branches of mathematics such as Analysis, Geometry, and Topology, and whose studies date back to the 19th century. Fixed point theory emerged to show the existence and uniqueness of the solution of ordinary differential equations. This theory plays an important role in solving problems encountered in many areas such as functional analysis, approximation theory, variational and linear inequalities, control systems, and optimization. In addition to its direct use in mathematics, fixed point theory has applications in various disciplines such as physics, chemistry, biology, medicine, engineering, communication, and economics [1-7].

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Fixed point theory studies are carried out on many mathematical structures from normed spaces to metric spaces. Studies on fixed point theory in normed spaces started with Brouwer [8] in the first quarter of the 20th century. Brouwer showed that mapping on itself defined on a closed ball in $\mathbb{R}^n$ space has at least one fixed point. In 1930, Schauder [9] took a Banach space instead of $\mathbb{R}^n$ space in Brouwer fixed point theorem and proved that a mapping on itself defined in a compact and convex subset of this space has at least one fixed point and presented a generalization of Brouwer fixed point theorem.

Fixed point theory studies on complete metric spaces started with Stephan Banach. In 1922, Banach [10] proved that in the Banach fixed point theorem, also known as the "contraction principle", which guarantees the existence and uniqueness of a fixed point, a contraction mapping defined in the complete metric space has a unique fixed point. This theorem also provides a method for calculating the uniquely determined fixed point of a contraction mapping. This method, called iteration, produces a sequence with sequential approaches, and this sequence converges to the unique fixed point of the contraction mapping. Through this approach, Banach fixed point theorem is used as an effective tool in solving many existence problems in mathematics.

Although iterative algorithms have a wide and dynamic literature, studies for these algorithms can be listed as determining the conditions under which the sequence to be obtained by using certain mapping classes will converge, comparing the convergence speeds, and performing data dependence and stability analysis [11-15].

The concept of stability has been widely used in many branches of science for a long time. According to Magnus [16], the source of stability studies can be found in the works of Aristoteles and Archimedes. At the same time, there is no universal definition of stability that is tailored to the specific needs of a particular problem. As a result, stability is one of the very meaningful scientific terms. In a broad sense, stability is understood as the ability of a system to continue functioning despite external disturbances without changing the internal structure.

The concept of stability of mathematical problems can be explained as follows: In what situations can we say that by making a small change in the hypotheses of a theorem, the main conclusion of the theorem remains true or approximately true?. The first study on stability in the sense of fixed point was presented by Urabe in 1956 [17]. In 1967, Ostrowski [18] studied the stability of Picard iterative algorithm in metric spaces, but in general terms, the concept of stability for iterative algorithms was defined by Harder and Hicks [19] in 1988 and they proved some stability theorems for Picard and Mann iterative algorithms using certain mapping classes. Later, this concept was examined by many authors and a large literature was created [20, 21].

In this study, a new four-step iterative algorithm has been proposed and the convergence of the newly defined iterative algorithm in a Banach space has been examined under suitable conditions. In addition, the convergence rate of the new algorithm has been compared with the existing algorithms and this result has been evaluated numerically on a non-trivial example. Finally, the concept of stability is analyzed for the new algorithm.

2. MATERIALS AND METHODS

In this section, we recall some fixed point iterative algorithms and give some preliminary information we need to obtain our main results.

Let X be an ambient space, $T: X \to X$ be a mapping, and $x_0, u_0 \in X$.

Algorithm 1.

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = T[\left(1-\alpha_n^{(1)}\right)x_n + \alpha_n^{(1)} z_n] \\ z_n = \left(1-\alpha_n^{(2)}\right)x_n + \alpha_n^{(2)} Tx_n \end{cases} \tag{1}$$

in which $\alpha_n^{(k)} \in [0,1]$ are control sequences for $k = 1, 2$. This method is called Thakur iterative algorithm [22].

Algorithm 2.

$$\begin{cases} u_{n+1} = Tv_n \\ v_n = T[\left(1-\alpha_n^{(1)}\right)w_n + \alpha_n^{(1)} Tw_n] \\ w_n = \left(1-\alpha_n^{(2)}\right)u_n + \alpha_n^{(2)} Tu_n \end{cases} \tag{2}$$

in which $\alpha_n^{(k)} \in [0,1]$ are control sequences for $k = 1,2$. This method is called K^* iterative algorithm defined by Ullah and Arshad [23].

Algorithm 3.

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = T[\left(1-\alpha_n^{(1)}\right)Tx_n + \alpha_n^{(1)} Tz_n] \\ z_n = \left(1-\alpha_n^{(2)}\right)x_n + \alpha_n^{(2)} Tx_n \end{cases} \tag{3}$$

in which $\alpha_n^{(k)} \in [0,1]$ are control sequences for $k = 1,2$. This method is called K iterative algorithm defined by Hussain et al. [24].

Algorithm 4.

$$\begin{cases} x_{n+1} = T[\left(1-\alpha_n^{(1)}\right)Tx_n + \alpha_n^{(1)} Ty_n] \\ y_n = T[\left(1-\alpha_n^{(2)}\right)x_n + \alpha_n^{(2)} Tx_n] \end{cases} \tag{4}$$

in which $\alpha_n^{(k)} \in [0,1]$ are control sequences for $k = 1,2$. This method is called Vatan-two step iterative algorithm defined by Vatan et al. [25].

Especially in applied mathematics, besides obtaining the solution of a problem, it is important to determine how to reach this solution in the easiest and fastest way. For this reason, it is necessary to compare the speed of an iterative algorithm used in the solution of the problem in comparison to other algorithms created with the same mapping. Based on this information, the four-step fixed-point iterative algorithm we defined is as follows:

Algorithm 5.

$$\begin{cases} x_{n+1} = T\left[\left(1-\alpha_n^{(1)}\right)Ty_n + \alpha_n^{(1)}Tz_n\right] \\ y_n = T\left[\left(1-\alpha_n^{(2)}\right)Tx_n + \alpha_n^{(2)}Tz_n\right] \\ z_n = T\left[\left(1-\alpha_n^{(3)}\right)w_n + \alpha_n^{(3)}Tw_n\right] \\ w_n = T\left[\left(1-\alpha_n^{(4)}\right)x_n + \alpha_n^{(4)}Tx_n\right] \end{cases} \tag{5}$$

in which $\alpha_n^{(k)} \in [0,1]$ are control sequences for $k = 1,2,3,4$.

Definition 1. Let X be a Banach space and $T: X \to X$ be a mapping. If there exist a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \phi(\|x - Tx\|) + \delta\|x - y\| \tag{6}$$

for all $x, y \in X$. Then, T is called contractive-like mapping [26].

This mapping may not always have a fixed point. However, if this mapping has a fixed point such as p, using the inequality (6), we get

$$\|Tx - p\| \leq \delta\|x - p\| \tag{7}$$

and using the inequality (7), we obtain

$$\begin{aligned} \|Tx - Ty\| &\leq \|Tx - p\| + \|p - Ty\| \\ &\leq \delta\|x - p\| + \delta\|y - p\| \\ &\leq \delta\|x - y\| + 2\delta\|y - p\| \end{aligned} \tag{8}$$

Lemma 1. Let $\left\{\alpha_n^{(k)}\right\}_{n=0}^{\infty}$ be three sequences such that $\alpha_n^{(k)} \geq 0$ ($\forall\, n \in \mathbb{N}$) for $k = 1,2,3$. Assume that $\alpha_n^{(2)} = o\left(\alpha_n^{(3)}\right)$, $\sum_{n=1}^{\infty} \alpha_n^{(3)} = \infty$, and $\alpha_n^{(3)} \in (0,1)$ for all $n \geq n_0$. If $\alpha_{n+1}^{(1)} \leq \left(1-\alpha_n^{(3)}\right)\alpha_n^{(1)} + \alpha_n^{(2)}$, then $\lim_{n\to\infty} \alpha_n^{(1)} = 0$ [27].

Lemma 2. Let $\left\{\alpha_n^{(k)}\right\}_{n=0}^{\infty}$ be two sequences such that $\alpha_n^{(k)} \geq 0$ ($\forall\, n \in \mathbb{N}$) for $k = 1{,}2$. Assume that $\lim_{n\to\infty} \alpha_n^{(2)} = 0$ and $\mu \in (0,1)$. If $\alpha_{n+1}^{(1)} \leq \mu\alpha_n^{(1)} + \alpha_n^{(2)}$, then $\lim_{n\to\infty} \alpha_n^{(1)} = 0$ [28].

Definition 2. If

$$\lim_{n\to\infty} \frac{\left\|\Theta_n^{(1)} - \Theta\right\|}{\left\|\Theta_n^{(2)} - \Theta\right\|} = 0,$$

in which $\left\{\Theta_n^{(i)}\right\}_{n=0}^{\infty}$ are two sequences with $\lim_{n\to\infty} \Theta_n^{(i)} = \Theta$ ($i = 1{,}2$), then it is said that $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$ [29].

Definition 3. Let (X, d) be a metric space, $T: X \to X$ be a mapping with fixed point p, and $x_0 \in X$. Assume that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by the iterative algorithm $x_{n+1} = f(T, x_n)$ converges to p. Let $\{y_n\}_{n=1}^{\infty} \subset X$ be an arbitrary sequence and set $\varepsilon_n = d\big(y_{n+1}, f(T, y_n)\big),\ n = 0,1,2,\ldots$ Then the iterative algorithm $f(T, x_n)$ will be called stable (or T-stable) if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$ [30].

3. RESULTS AND DISCUSSION

In this section, strong convergence, rate of convergence, and stability results for the sequence obtained from Algorithm 5 constructed using a contractive-like mapping satisfying the condition (6), will be discussed.

Theorem 1. Let C be a nonempty, closed, convex subset of a Banach space X and $T: C \to C$ be a contractive-like mapping with fixed point p. Let $\{x_n\}_{n=1}^{\infty}$ be iterative sequence generated by Algorithm 5 with real sequence $\left\{\alpha_n^{(1)}\right\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n^{(1)} = \infty$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to p.

Proof. We have to show $x_n \to p$ as $n \to \infty$. By using Algorithm 5 and (6), we have

$$\begin{aligned}
\|w_n - p\| &= \left\|T[(1 - \alpha_n^{(4)})x_n + \alpha_n^{(4)}Tx_n] - p\right\| \\
&\leq \delta\left\|(1 - \alpha_n^{(4)})x_n + \alpha_n^{(4)}Tx_n - p\right\| \\
&\leq \delta(1 - \alpha_n^{(4)})\|x_n - p\| + \alpha_n^{(4)}\delta\|Tx_n - p\| \\
&\leq \delta(1 - \alpha_n^{(4)})\|x_n - p\| + \alpha_n^{(4)}\delta^2\|x_n - p\| \\
&\leq \delta[1 - \alpha_n^{(4)}(1 - \delta)]\|x_n - p\|
\end{aligned} \tag{9}$$

and

$$\begin{aligned}\|z_n - p\| &= \left\|T[(1-\alpha_n^{(3)})w_n + \alpha_n^{(3)}Tw_n] - p\right\| \\ &\leq \delta\left\|(1-\alpha_n^{(3)})w_n + \alpha_n^{(3)}Tw_n - p\right\| \\ &\leq \delta(1-\alpha_n^{(3)})\|w_n - p\| + \alpha_n^{(3)}\delta\|Tw_n - p\| \\ &\leq \delta(1-\alpha_n^{(3)})\|w_n - p\| + \alpha_n^{(3)}\delta^2\|w_n - p\| \\ &\leq \delta[1-\alpha_n^{(3)}(1-\delta)]\|w_n - p\|\end{aligned} \tag{10}$$

Substituting (9) in (10), we attain

$$\|z_n - p\| \leq \delta^2\left[1-\alpha_n^{(3)}(1-\delta)\right]\left[1-\alpha_n^{(4)}(1-\delta)\right]\|x_n - p\| \tag{11}$$

Also,

$$\begin{aligned}\|y_n - p\| &= \left\|T[(1-\alpha_n^{(2)})Tx_n + \alpha_n^{(2)}Tz_n] - p\right\| \\ &\leq \delta\left\|(1-\alpha_n^{(2)})Tx_n + \alpha_n^{(2)}Tz_n - p\right\| \\ &\leq \delta^2(1-\alpha_n^{(2)})\|x_n - p\| + \alpha_n^{(2)}\delta^2\|z_n - p\|\end{aligned} \tag{12}$$

Substituting (11) in (12), we get

$$\begin{aligned}\|y_n - p\| &\leq \delta^2\{(1-\alpha_n^{(2)})\|x_n - p\| \\ &\quad +\alpha_n^{(2)}\delta^2\left[1-\alpha_n^{(3)}(1-\delta)\right][1-\alpha_n^{(4)}(1-\delta)]\}\|x_n - p\|\end{aligned} \tag{13}$$

Moreover,

$$\begin{aligned}\|x_{n+1} - p\| &= \left\|T[(1-\alpha_n^{(1)})Ty_n + \alpha_n^{(1)}Tz_n] - p\right\| \\ &\leq \delta\left\|(1-\alpha_n^{(1)})Ty_n + \alpha_n^{(1)}Tz_n - p\right\| \\ &\leq \delta(1-\alpha_n^{(1)})\|Ty_n - p\| + \alpha_n^{(1)}\delta\|Tz_n - p\| \\ &\leq \delta^2(1-\alpha_n^{(1)})\|y_n - p\| + \alpha_n^{(1)}\delta^2\|z_n - p\|\end{aligned} \tag{14}$$

Substituting (11) and (13) in (14), we get

$$\begin{aligned}\|x_{n+1} - p\| &\leq \delta^3(1-\alpha_n^{(1)})\{(1-\alpha_n^{(2)})\delta\|x_n - p\| \\ &\quad +\alpha_n^{(2)}\delta^3\left[1-\alpha_n^{(3)}(1-\delta)\right][1-\alpha_n^{(4)}(1-\delta)]\}\|x_n - p\| \\ &\quad +\alpha_n^{(1)}\delta^4\left[1-\alpha_n^{(3)}(1-\delta)\right][1-\alpha_n^{(4)}(1-\delta]\|x_n - p\|\end{aligned}$$

Considering that $\alpha_n^{(k)} \in [0,1]$ $(k = 1,2,3,4)$, $\delta \in (0,1)$, and rearranging the above inequality, we obtain

$$\|x_{n+1} - p\| \leq \delta^3[1-\alpha_n^{(1)}(1-\delta]\|x_n - p\| \tag{15}$$

By induction, we get

$$\|x_n - p\| \leq \delta^3[1-\alpha_{n-1}^{(1)}(1-\delta]\|x_{n-1} - p\|$$

$$\|x_{n-1}-p\| \leq \delta^3[1-\alpha_{n-2}^{(1)}(1-\delta]\|x_{n-2}-p\|$$
$$\vdots$$
$$\|x_2-p\| \leq \delta^3[1-\alpha_1^{(1)}(1-\delta]\|x_1-p\|$$

Hence,

$$\|x_{n+1}-p\| \leq \|x_1-p\|\delta^{3n}\prod_{i=1}^{n}\left[1-\alpha_i^{(1)}(1-\delta)\right] \tag{16}$$

Since $\alpha_i^{(1)} \in [0,1]$ and $\delta \in (0,1)$, we have $[1-\alpha_i^{(1)}(1-\delta] \leq 1$. It is well-known from classical analysis that $1-x \leq e^{-x}$ for all $x \in [0,1]$, hence from (16), we attain

$$\|x_{n+1}-p\| \leq \|x_1-p\|\delta^{3n}\prod_{i=1}^{n} e^{-(1-\delta)\alpha_i^{(1)}}$$
$$= \|x_1-p\|\delta^{3n}e^{-(1-\delta)\sum_{i=1}^{n}\alpha_i^{(1)}}$$

Taking the limit of both sides of the above inequality, $x_n \to p$ as $n \to \infty$.

Without the need for the $\sum_{n=1}^{\infty}\alpha_n^{(1)} = \infty$ condition given in this theorem, it can be seen from the following theorem that $\lim_{n\to\infty}\|x_n-p\| = 0$:

Theorem 2. Let $C, X,$ and T with fixed point p be the same as in Theorem 1. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$, which is generated by Algorithm 5, converges strongly to p.

Proof. From (15), we have

$$\|x_{n+1}-p\| \leq \delta^3[1-\alpha_n^{(1)}(1-\delta]\|x_n-p\|$$

Since $[1-\alpha_n^{(1)}(1-\delta] \leq 1$, we get

$$\|x_{n+1}-p\| \leq \delta^3\|x_n-p\|$$

If the limit is taken for this inequality, it can be easily seen that $\lim_{n\to\infty}\|x_n-p\| = 0$ since $\delta \in (0,1)$.

Theorem 3. Let $C, X,$ and T with fixed point p be the same as in Theorem 1. For given $x_1 = u_1 \in C$, consider the iterative sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ defined by Algorithm 5 and Algorithm 2, respectively with $\left\{\alpha_n^{(k)}\right\}_{n=0}^{\infty} \in [0,1]$ for $k = 1,2,3,4$ in which $\alpha_1^{(1)} \leq \alpha_n^{(1)} \leq 1$. Then $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{u_n\}_{n=1}^{\infty}$ to p.

Proof. From Theorem 1, we have the following estimate:

$$\|x_{n+1}-p\| \leq \|x_1-p\|\delta^{3n}\prod_{i=1}^{n}\left[1-\alpha_i^{(1)}(1-\delta)\right] \tag{17}$$

Also, by using Algorithm 2, we get

$$\begin{aligned}\|w_n - p\| &\le \left(1-\alpha_n^{(2)}\right)\|u_n - p\| + \delta\alpha_n^{(2)}\|u_n - p\| \\ &= \left[1-\alpha_n^{(2)}(1-\delta)\right]\|u_n - p\| \end{aligned} \tag{18}$$

and

$$\begin{aligned}\|v_n - p\| &\le \delta\left(1-\alpha_n^{(1)}\right)\|w_n - p\| + \delta\alpha_n^{(1)}\|w_n - p\| \\ &\le \delta\left[1-\alpha_n^{(1)}(1-\delta)\right]\|w_n - p\| \end{aligned} \tag{19}$$

and

$$\|u_{n+1} - p\| \le \delta\|v_n - p\| \tag{20}$$

Substituting (18) in (19) and (19) in (20), respectively, and by taking into account that $\left[1-\alpha_n^{(2)}(1-\delta)\right] \le 1$, we attain

$$\|u_{n+1} - p\| \le \delta^2\left[1-\alpha_n^{(1)}(1-\delta)\right]\|u_n - p\| \tag{21}$$

By induction, we have

$$\|u_{n+1} - p\| \le \delta^{2n}\prod_{i=1}^{n}\left[1-\alpha_i^{(1)}(1-\delta)\right]\|u_1 - p\| \tag{22}$$

By applying assumption $\alpha_1^{(1)} \le \alpha_n^{(1)} \le 1$ to (17) and (22) respectively, we obtain

$$\begin{aligned}\|x_{n+1} - p\| &\le \delta^{3n}\prod_{i=1}^{n}\left[1-\alpha_1^{(1)}(1-\delta)\right]\|x_1 - p\| \\ &= \delta^{3n}\left[1-\alpha_1^{(1)}(1-\delta)\right]^n\|x_1 - p\| \end{aligned} \tag{23}$$

and

$$\begin{aligned}\|u_{n+1} - p\| &\le \delta^{2n}\prod_{i=1}^{n}\left[1-\alpha_1^{(1)}(1-\delta)\right]\|u_1 - p\| \\ &= \delta^{2n}\left[1-\alpha_1^{(1)}(1-\delta)\right]^n\|u_1 - p\| \end{aligned} \tag{24}$$

From (23) and (24), we can choose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as

$$\begin{aligned} a_n &= \delta^{3n}[1-\alpha_1^{(1)}(1-\delta)]^n\|x_1 - p\| \\ b_n &= \delta^{2n}[1-\alpha_1^{(1)}(1-\delta)]^n\|u_1 - p\| \end{aligned} \tag{25}$$

respectively. Let $\Theta_n = \frac{a_n}{b_n}$. Then, we have

$$\Theta_n = \frac{\|x_1 - p\|\delta^{3n}[1-\alpha_1^{(1)}(1-\delta)]^n}{\|u_1 - p\|\delta^{2n}[1-\alpha_1^{(1)}(1-\delta)]^n} = \delta^n$$

Since $\delta \in (0,1)$, we obtain $\lim_{n\to\infty} \Theta_n = 0$ which implies that $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{u_n\}_{n=1}^{\infty}$.

In order to validity of Theorem 3, we give the following example:

Example 3.1. Let $X = \mathbb{R}$ and $C = [0,1]$. Let $T: C \to C$ be a mapping defined by $T(x) = \frac{1}{5}e^{-2x} + \frac{1}{3}cos2x$ for all $x \in C$. It can be seen in the Fig. 1 that T satisfies condition (6) with $\delta = 0.75$ and $\psi(t) = \frac{3t}{25}$:

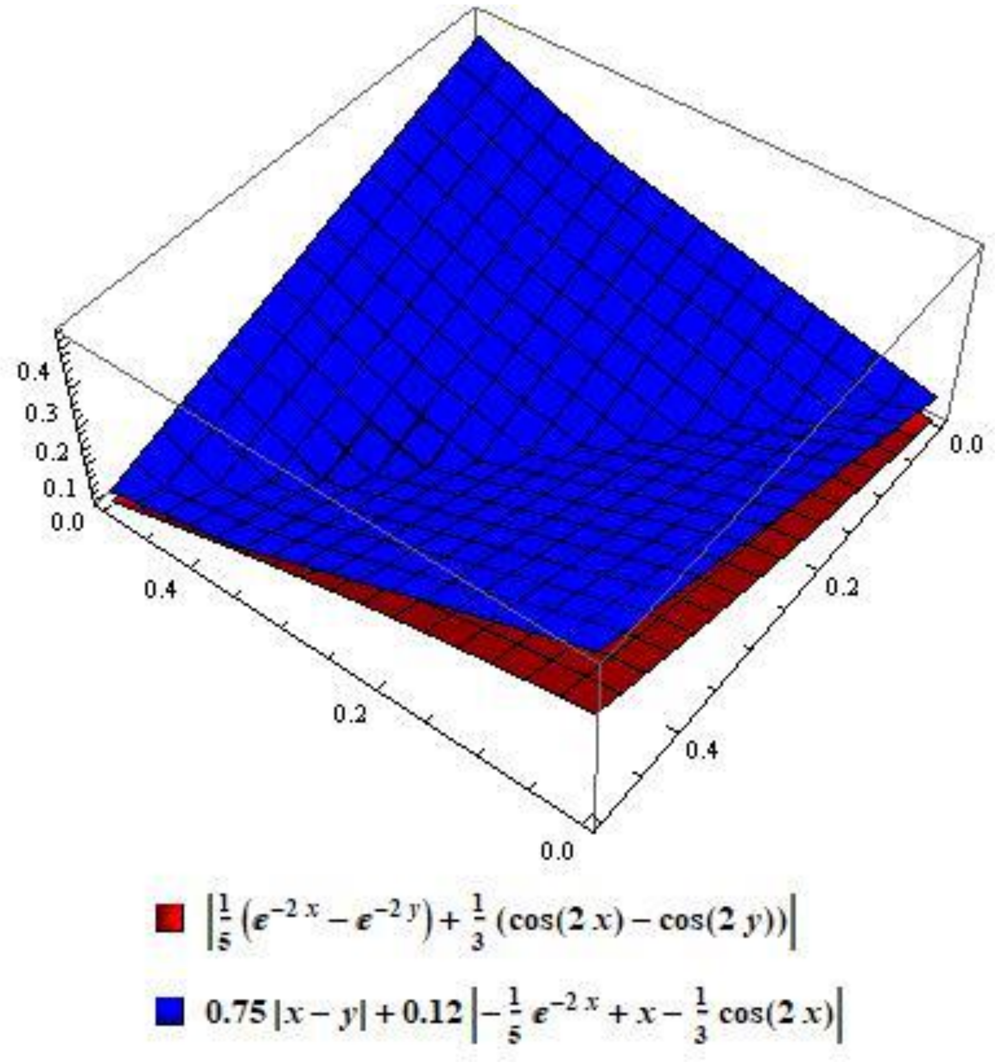

Figure 1. Graphical demonstration of T

Let $x_1 = 0.99$ and $\alpha_n^{(k)} = 0.25$ for $k = 1,2,3,4$. The following table shows that Algorithm 5 converges to $p = 0.35261795533135$ faster than all Algorithms 1-4:

x_n	Algorithm 5	Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 4
x_1	0.99	0.99	0.99	0.99	0.99
x_2	0.41706233263467	0.55568990364746	0.43575190477400	0.21435139142803	0.44444626305663
…	…	…	…	…	…
x_{15}	0.35261795533136	0.35261826270570	0.35261795533195	0.35261795581020	0.35261795533203
x_{16}	0.35261795533135	0.35261806481107	0.35261795533143	0.35261795522394	0.35261795533144
x_{17}	…	0.35261799432552	0.35261795533136	0.35261795535544	0.35261795533136
x_{18}		0.35261796922018	0.35261795533135	0.35261795532594	0.35261795533135
…		…	…	…	…
x_{23}		0.35261795541096		0.35261795533135	
		…		…	
x_{33}		0.35261795533135			

Theorem 4. Let C, X, and T with fixed point p be the same as in Theorem 1. Suppose that iterative sequence generated by Algorithm 5 converges to p. Then Algorithm 5 is T −stable.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X. Define

$$\varepsilon_n = \|a_{n+1} - f(T, a_n)\|$$

for all $n \in \mathbb{N}$, in which $a_{n+1} = T\left[\left(1-\alpha_n^{(1)}\right)Tb_n + \alpha_n^{(1)}Tc_n\right]$, $b_n = T\left[\left(1-\alpha_n^{(2)}\right)Ta_n + \alpha_n^{(2)}Tc_n\right]$, $c_n = T\left[\left(1-\alpha_n^{(3)}\right)d_n + \alpha_n^{(3)}Td_n\right]$and $d_n = T\left[\left(1-\alpha_n^{(4)}\right)a_n + \alpha_n^{(4)}Ta_n\right]$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. We prove that $\lim_{n\to\infty} a_n = p$:

$$\begin{aligned}\|a_{n+1} - p\| &\le \|a_{n+1} - f(T, a_n)\| \\ &\quad + \|f(T, a_n) - p\| \\ &\le \varepsilon_n + \delta\left\|\left(1-\alpha_n^{(1)}\right)Tb_n + \alpha_n^{(1)}Tc_n - p\right\| \\ &\le \varepsilon_n + \delta\left(1-\alpha_n^{(1)}\right)\|Tb_n - p\| + \delta\alpha_n^{(1)}\|Tc_n - p\| \\ &\le \varepsilon_n + \delta^2\left(1-\alpha_n^{(1)}\right)\|b_n - p\| + \delta^2\alpha_n^{(1)}\|c_n - p\|\end{aligned} \tag{26}$$

and

$$\begin{aligned}\|b_n - p\| &= \left\|T[(1-\alpha_n^{(2)})Ta_n + \alpha_n^{(2)}Tc_n] - p\right\| \\ &\le \delta\left\|(1-\alpha_n^{(2)})Ta_n + \alpha_n^{(2)}Tc_n - p\right\| \\ &\le \delta^2(1-\alpha_n^{(2)})\|a_n - p\| + \alpha_n^{(2)}\delta^2\|c_n - p\|\end{aligned} \tag{27}$$

and

$$\begin{aligned}\|c_n - p\| &= \left\|T[(1-\alpha_n^{(3)})d_n + \alpha_n^{(3)}Td_n] - p\right\| \\ &\le \delta(1-\alpha_n^{(3)})\|d_n - p\| + \alpha_n^{(3)}\delta^2\|d_n - p\| \\ &\le \delta[1-\alpha_n^{(3)}(1-\delta)]\|d_n - p\|\end{aligned} \tag{28}$$

and

$$\begin{aligned}\|d_n - p\| &= \left\|T[(1-\alpha_n^{(4)})a_n + \alpha_n^{(4)}Ta_n] - p\right\| \\ &\le \delta(1-\alpha_n^{(4)})\|a_n - p\| + \alpha_n^{(4)}\delta^2\|a_n - p\| \\ &\le \delta[1-\alpha_n^{(4)}(1-\delta)]\|a_n - p\|\end{aligned} \tag{29}$$

Substituting (29) in (28), we attain

$$\|c_n - p\| \le \delta^2\left[1-\alpha_n^{(3)}(1-\delta)\right]\left[1-\alpha_n^{(4)}(1-\delta)\right]\|a_n - p\| \tag{30}$$

Substituting (30) in (27), we get

$$\|b_n - p\| \leq \delta^2 \left\{1 - \alpha_n^{(2)}(1-\delta^2)\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right]\right\} \times \|a_n - p\| \tag{31}$$

Moreover, substituting (31) and (30) in (26), we obtain

$$\|a_{n+1} - p\| \leq \varepsilon_n + \delta^4\{1 - \alpha_n^{(2)}(1-\delta^2)\left(1-\alpha_n^{(1)}\right)\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right] + \alpha_n^{(1)}\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right]\}\|a_n - p\|$$

Since $\delta \in [0,1)$ and $\left\{\alpha_n^{(k)}\right\}_{n=0}^{\infty} \in [0,1]$ for $k = 1,2,3,4$, we have

$$\delta^4\{1 - \alpha_n^{(2)}(1-\delta^2)\left(1-\alpha_n^{(1)}\right)\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right] + \alpha_n^{(1)}\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right]\} < 1$$

Hence,

$$\|a_{n+1} - p\| \leq \delta\|a_n - p\| + \varepsilon_n$$

Thus, from Lemma 2 we obtain $\lim_{n\to\infty} a_n = p$.

Conversely, assume that $\lim_{n\to\infty} a_n = p$. We now prove that $\lim_{n\to\infty} \varepsilon_n = 0$:

$$\begin{aligned} \varepsilon_n &= \|a_{n+1} - f(T, a_n)\| \\ &\leq \|a_{n+1} - p\| + \left\|T\left[\left(1-\alpha_n^{(1)}\right)Tb_n + \alpha_n^{(1)}Tc_n\right] - p\right\| \\ &\leq \|a_{n+1} - p\| + \delta\left\|\left(1-\alpha_n^{(1)}\right)Tb_n + \alpha_n^{(1)}Tc_n - p\right\| \\ &\leq \|a_{n+1} - p\| + \delta^2\left(1-\alpha_n^{(1)}\right)\|b_n - p\| + \delta^2\alpha_n^{(1)}\|c_n - p\| \end{aligned} \tag{32}$$

Substituting (30) and (31) in (32), we get

$$\varepsilon_n \leq \|a_{n+1} - p\| + \delta^4\{1 - \alpha_n^{(2)}(1-\delta^2)\left(1-\alpha_n^{(1)}\right)\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right] + \alpha_n\left[1 - \alpha_n^{(3)}(1-\delta)\right]\left[1 - \alpha_n^{(4)}(1-\delta)\right]\}\|a_n - p\|$$

By taking the limit as $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} \varepsilon_n = 0$.

4. CONCLUSIONS

In this study, we analyzed the convergence of Algorithm 5 under appropriate conditions. In addition to proving that it has a better convergence speed than other algorithms in the literature, we also presented a numerical example to support this result. Finally, we discussed the concept of stability for Algorithm 5.

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