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Research Article

Quantum Mechanics Approach for Appropriately Chosen Hamiltonian

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Abstract

Risk theory has always played a significant role in mathematical finance and actuarial sciences. A novel approach to the risk theory of non-life insurance is quantum mechanics. To compute finite-time non-ruin probability, I introduce the quantum mechanics formalism in discrete space and continuous space with the appropriately chosen Hamiltonian. By using the quantum mechanics approach and the stochastic method, the non-ruin operator is defined, and tensor products of operator concepts are presented for several examples.

In this paper, Dirac notations are operated to find the Hamiltonian matrix with the eigenvector basis for two and three-state cases, and its tensor product version with a change of basis.

Keywords
Ruin Probability, Risk Theory, Hamiltonian, Quantum Mechanics, Dirac Notations, Tensor Product, Non-life Insurance.

1. INTRODUCTION

The risk theory, which can be found in numerous works such as Asmussen and Albrecher [1] and Gerber [2], is one of the most important arguments in the world of actuarial science [3]. In this paper, the representation of quantum mechanics, which depends on the Dirac formalism [4], is introduced. In a range of scientific areas, quantum mechanics represents a promising new method. Baaquie [5-6], especially studies on the financial perspective of the quantum approach and the mathematical background of quantum mechanics. In addition to this, Griffiths [7] explains how quantum mechanics works and how to treat perturbation theory by using it elaboratively.

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Quantum mechanics is a well-known branch of physics that describes the behavior and random evolution of matter or light on an atomic scale [8-9]. Furthermore, the Hamiltonian is the corresponding operator in quantum mechanics and represents the total energy of the system [7]. This novel approach has been successfully applied in various fields and is an efficient alternative tool to classical probability calculations. In this paper, the quantum approach to non-life insurance is discussed, non-ruin probability is modeled, and several advanced examples are treated for chosen Hamiltonians in discrete and continuous space with traditional basis and eigenvalues of Hermitian operators in two and three dimensions by using the tensor product of operators and standard Dirac matrix formalism [4] with bra-ket notations.

In Section 2, the Hamiltonian approach is introduced, and the tensor product of an operator is defined to find the Hamiltonian matrix with the eigenvector basis. In Section 3, several examples are presented, and Dirac notations are operated to find the Hamiltonian matrix with the eigenvector basis for two and three-state cases, and its tensor product version with change of basis.

2. HAMILTONIAN APPROACH

In this section, classical risk theory is defined, and the Hamiltonian approach is introduced by using Dirac notations and tensor products.

2.1. Classical Risk Theory

Define the classical surplus process as [10-13]

$$R(t) = X_0 + ct - S(t) \quad ; \quad t \geq 0 \tag{1}$$

where X_0 is the initial capital, c is the constant premium rate and

$$S(t) = \sum_{j=1}^{N_t} X_j \tag{2}$$

is the total claim amount where claim X_j are i.i.d random variables and independent of a Poisson Process N_t. Also, Markov chains are a significant and more practical type of random process since their richness and practicability are sufficient to serve a diverse range of applications [14-16]. Based on the Markov property, one-step transition probabilities and n-step transition probabilities can easily be defined.

2.2. Hamiltonian Operator

Let H be a Hamiltonian matrix operator and A_t be a semi-group. A unique operator can be derived as

$$< x|A_t|x' >= \sum_i < x|A_t|i >< i|x' > \tag{3}$$

for any bra vector $< x|$ and ket vector $|x' >$ in H. By substituting $\exp(-tH)$ into the semi-group A_t, the following Dirac notation is obtained as [17-18]

$$< x|\exp(-tH)|x' >= \sum_i < x|\exp(-tH)|i >< i|x' >. \quad (4)$$

Hamiltonian is operated to calculate the kernel [5]

$$P(x, x'; \zeta) =< x|\exp(-\zeta H)|x' > \quad (5)$$

where $\zeta = \mathrm{T} - \mathrm{t}$. By using delta function, I find [5]

$$< x|x' >= \delta(x - x') = \int_{-\infty}^{\infty} \frac{dp}{2\pi} exp\big(ip(x - x')\big) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|p >< p|x' >. \quad (6)$$

For basis $|p >$ in the completeness equation [5], the following is shown as

$$\int_{-\infty}^{\infty} \frac{dp}{2p} |p >< p| = I \quad (7)$$

with the scalar product $< x|p >= \exp(ipx)$ and $< p|x >= \exp(-ipx)$. Hence [5]

$$P(x, x'; \zeta) =< x|\exp(-\zeta H)|x' >= \int_{-\infty}^{\infty} \frac{dp}{2\pi} exp(-\zeta H)\,|p >< p|x' >. \quad (8)$$

Then, consider H is applied to a ket vector $|i >$ such that $H|i >= \delta_i|i >$ and semi-group A_t is applied to a ket vector $|i >$, I get the following equations

$$A_t|i >= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} H^k|i >= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} {\delta_i}^k|i >= \exp(-t\delta_i)|i > . \quad (9)$$

Hence, Eq. (4) is concluded as

$$< x|\exp(-tH)|x' >= \sum_i < x|i >< i|x' > \exp(-t\delta_i). \quad (10)$$

2.3. Tensor Product of an Operator

Let x and y be two initial states and x' and y' be the final points, respectively. The operator is denoted by [6, 19]

$$< x\otimes y|A\otimes B|x'\otimes y' >= (x\otimes y)^T(A\otimes B)(x'\otimes y') = (x^T\otimes y^T)(A\otimes B)(x'\otimes y') \quad (11)$$

where T means transpose. By the property of tensor product, the following property can be reached [6, 19]

$$< x\otimes y|A\otimes B|x'\otimes y' >= [x^T A\otimes y^T B](x'\otimes y') = (x^T Ax'\otimes y^T By'). \quad (12)$$

Then, Eq. (11) can be written as [6]

$$< x\otimes y|A\otimes B|x'\otimes y' >=< x|A|x' > \otimes < y|B|y' >. \quad (13)$$

Here, we know that $< x|A|x' >$ and $< y|B|y' >$ are numbers. So, the operator is derived as

$$< x\otimes y|A\otimes B|x'\otimes y' >=< x|A|x' >< y|B|y' >. \quad (14)$$

Fact 1: Let A, B, C and D be matrix. Multiplication of tensor products $A\otimes B$ and $C\otimes D$ is denoted by

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD). \quad (15)$$

Fact 2: If the notation $Ax = A|x >$, then

$$(A\otimes B)|C\otimes D >= (A|C >)\otimes(B|D >). \quad (16)$$

Now, if we generalize the system for initial states $\{x_1, x_2, \dots, x_n\}$, final points $\{x_1', x_2', \dots, x_n'\}$, and set of matrices $\{A_1, A_2, \dots, A_n\}$, the operator is defined by

$$< x_1 \otimes x_2 \otimes \dots \otimes x_n | A_1 \otimes A_2 \otimes \dots \otimes A_n | x_1' \otimes x_2' \otimes \dots \otimes x_n' > = < x_1 | A_1 | x_1' > \otimes < x_2 | A_2 | x_2' > \otimes \dots \dots \otimes < x_n | A_n | x_n' >. \quad (17)$$

Simply, it can be written as

$$\prod_{i=1}^{n} < x_i | A_i | x_i' >. \quad (18)$$

Lemma 2.3.1: Let $K_i{}^A$ and $K_j{}^B$ be the eigenvalues for matrix A and B, respectively. If $H_{A\otimes B} = I_A \otimes H_B + H_A \otimes I_B$, I get the following equation

$$H_{A\otimes B} | i \otimes j >= K_\sigma | \sigma > \quad (19)$$

where $K_\sigma = K_j{}^B + K_i{}^A$ and $\sigma = | i \otimes j >= | i > \otimes | j >$.

Proof: If $K_i{}^A$ and $K_j{}^B$ are eigenvalues, it can be said that $H_A \big| i >= K_i{}^A \big| i >$ and $H_B \big| j >= K_j{}^B \big| j >$. Using the notations $H_{A\otimes B} = I_A \otimes H_B + H_A \otimes I_B$, the following notation is derived as

$$H_{A\otimes B} | i \otimes j >= I_A \otimes H_B | i \otimes j > + H_A \otimes I_B | i \otimes j >. \quad (20)$$

From the properties of the tensor product, I get

$$H_{A\otimes B} | i \otimes j >= I_A | i > \otimes H_B | j > + H_A | i > \otimes I_B | j >= | i > \otimes K_j{}^B \big| j > + K_i{}^A \big| i > \otimes | j >. \quad (21)$$

Hence,

$$H_{A\otimes B} | i \otimes j >= (K_j{}^B + K_i{}^A) | i \otimes j >= K_\sigma | \sigma >. \quad (22)$$

Lemma 2.3.2: Let $K_i{}^A$, $K_j{}^B$ and $K_k{}^C$ be the eigenvalues for matrix A, B and C, respectively. If $H_{A\otimes B\otimes C} = H_A \otimes I_B \otimes I_C + I_A \otimes H_B \otimes I_C + I_A \otimes I_B \otimes H_C$, the following notation is reached as

$$H_{A\otimes B\otimes C} | i \otimes j \otimes k >= K_\sigma | \sigma > \quad (23)$$

where $K_\sigma = K_i{}^A + K_j{}^B + K_k{}^C$ and $\sigma = | i \otimes j \otimes k >= | i > \otimes | j > \otimes | k >$.

Proof: A similar procedure is applied. If $K_i{}^A$, $K_j{}^B$ and $K_k{}^C$ are eigenvalues, it can be said that $H_A \big| i >= K_i{}^A \big| i >$, $H_B \big| j >= K_j{}^B \big| j >$ and $H_C \big| k >= K_k{}^C \big| k >$. Using the notations $H_{A\otimes B\otimes C} = H_A \otimes I_B \otimes I_C + I_A \otimes H_B \otimes I_C + I_A \otimes I_B \otimes H_C$, the following notation is derived as

$$H_{A\otimes B\otimes C} | i \otimes j \otimes k >= H_A \otimes I_B \otimes I_C | i \otimes j \otimes k > + I_A \otimes H_B \otimes I_C | i \otimes j \otimes j > + I_A \otimes I_B \otimes H_C | i \otimes j \otimes j > \quad (24)$$

From the properties of the tensor product, I get

$$H_{A\otimes B\otimes C} | i \otimes j \otimes k >= H_A | i > \otimes I_B | j > \otimes I_C | k > + I_A | i > \otimes H_B | j > \otimes I_C | k > + I_A | i > \otimes I_B | j > \otimes H_C | k >= H_A | i > \otimes | j > \otimes | k > + | i > \otimes H_B | j > \otimes | k > + | i > \otimes | j > \otimes H_C | k > \quad (25)$$

If I substitute the eigenvalues, Eq. (25) is written as

$$H_{A\otimes B\otimes C}|i\otimes j\otimes k>=K_i^A|\mathrm{i}>\otimes|j>\otimes|k>+|i>\otimes K_j^B|j>\otimes|k>+|i>\otimes|j>\otimes K_k^C|k>. \tag{26}$$

Hence,

$$H_{A\otimes B\otimes C}|i\otimes j\otimes k>=(K_i{}^A+K_j{}^B+K_k{}^C)|i\otimes j\otimes k>=K_\sigma|\sigma>. \tag{27}$$

Fact 3: By induction, I generalize the previous lemmas as follows:

$$H_A|i_1\otimes\ldots\otimes i_n>=(\textstyle\sum_{j=1}^n K_{i_j}^{A_j})|i_1\otimes\ldots\otimes i_n>=K_\sigma|\sigma> \tag{28}$$

where $H_A=H_{(A_1\otimes\ldots\otimes A_n)}=\sum_{j=1}^n I_{A_1}\otimes\ldots\otimes I_{A_{j-1}}H_{A_j}I_{A_{j+1}}\otimes\ldots\otimes I_{A_n}$ and $H_{A_j}|i_j>=K_{i_j}^{A_j}|i_j>$.

Furthermore, the operator notation is derived as

$$<x|\exp(-tH)|x'>=\textstyle\sum_\sigma <x|\exp(-tH)|\sigma><\sigma|x'> \tag{29}$$

where $x=x_1\otimes\ldots\otimes x_n$ are initial states, $x'=x_1{}'\otimes\ldots\otimes x_n{}'$ are final points and $\mathrm{H}|\sigma>=K|\sigma>$. So, I get

$$<x|\exp(-tH)|x'>=\textstyle\sum_\sigma \exp(-tK_\sigma)<x|\sigma><\sigma|x'>. \tag{30}$$

3. RUIN PROBABILITY EXAMPLES

Example 3.1: Let $H=\begin{pmatrix}0&0&0\\0&-(\lambda+\mu)&\lambda\\0&\mu&-2\mu\end{pmatrix}\in R^3$ be a Hamiltonian matrix and $A_\zeta=\exp(-\zeta H)$ be a generated semi-group. Then, the non-ruin probability matrix is represented as

$$\begin{pmatrix}1&0&0\\0&\frac{\exp(\zeta\mu)}{\Delta}(\mu+\lambda\exp(\zeta\Delta))&\frac{\exp(\zeta\mu)}{\Delta}(1-\exp(\zeta\Delta))\\0&\frac{\mu\exp(\zeta\mu)}{\Delta}(1-\exp(\zeta\Delta))&\frac{\exp(\zeta\mu)}{\Delta}(\lambda+\mu\exp(\zeta\Delta))\end{pmatrix} \tag{31}$$

where $\Delta=\lambda+\mu$.

Solution: Let $|1^*>=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $|2^*>=\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $|3^*>=\begin{pmatrix}0\\\lambda\\-\mu\end{pmatrix}$ be ket vectors. The (normalized) eigenvectors are simply represented as unit vectors $|1>=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $|2>=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $|3>=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

Then, I reach that

$(1\quad 0\quad 0)^T=|1>=|1^*>$,

$$(0 \quad 1 \quad 0)^T = |2> = \frac{\mu}{\Delta}|2^*> + \frac{1}{\Delta}|3^*>, \tag{32}$$

$$(0 \quad 0 \quad 1)^T = |3> = \frac{\lambda}{\Delta}|2^*> - \frac{1}{\Delta}|3^*>$$

where $\Delta = \lambda + \mu$.

A specific matrix is used to have diagonal form by change of basis, and it is called diagonalizable [7]. To change the coordinates of vectors, I take matrix Σ

$$\Sigma = A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \lambda & -\mu \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 1 & -\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda-\mu \\ 0 & \lambda-\mu & \lambda^2+\mu^2 \end{pmatrix} \tag{33}$$

where A^T consists of the ket vectors $|1^*>$, $|2^*>$ and $|3^*>$. By applying ket vectors $|1>$, $|2>$ and $|3>$ to the matrix Σ, I obtain

$$\Sigma 1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda-\mu \\ 0 & \lambda-\mu & \lambda^2+\mu^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1^*> \ ,$$

$$\Sigma 2 = \frac{1}{\Delta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda-\mu \\ 0 & \lambda-\mu & \lambda^2+\mu^2 \end{pmatrix}\begin{pmatrix} 0 \\ \mu \\ 1 \end{pmatrix} = |2^*> + \lambda|3^*> \ ,$$

$$\Sigma 3 = \frac{1}{\Delta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda-\mu \\ 0 & \lambda-\mu & \lambda^2+\mu^2 \end{pmatrix}\begin{pmatrix} 0 \\ \lambda \\ -1 \end{pmatrix} = |2^*> - \mu|3^*>.$$

Then, the operator in the new coordinates is defined as

$$\begin{aligned} < x|\exp(-\zeta H)|x'> &= < x|\exp(-\zeta H)|\Sigma x'>_* \\ &= < \Sigma x|\exp(-\zeta H)|1^*>_* < 1^*|x'>_* + < \Sigma x|\exp(-\zeta H)|2^*>_* < 2^*|x'>_* \\ &\quad + < \Sigma x|\exp(-\zeta H)|3^*>_* < 3^*|x'>_*. \end{aligned} \tag{34}$$

Now, if I apply eigenvectors to the matrix H, I find

$$\begin{aligned} H|1^*> &= 0|1^*> &\rightarrow\quad &\exp(-\zeta H)|1^*> = |1^*> \\ H|2^*> &= -\mu|2^*> &\rightarrow\quad &\exp(-\zeta H)|2^*> = \exp(-\zeta\mu)|2^*> \\ H|3^*> &= -(\lambda+2\mu)|3^*> &\rightarrow\quad &\exp(-\zeta H)|3^*> = \exp(-\zeta(\lambda+2\mu))|3^*> \end{aligned} \tag{35}$$

The operator is now derived as

$$< x|\exp(-\zeta H)|x'> = < \Sigma x|1^*>_* < 1^*|x'>_* + \exp(-\zeta\mu) \ < \Sigma x|2^*>_* < 2^*|x'>_* + \exp(-\zeta(\lambda+2\mu)) < \Sigma x|3^*>_* < 3^*|x'>_* \ . \tag{36}$$

So, the probabilities are found as follows:

$$\begin{aligned} P_{11}(\zeta) &= < 1|A_\zeta|1> \\ &= [< 1^*|1^*>< 1^*|1^*>] + \exp(-\zeta\mu)[< 1^*|2^*>< 2^*|1^*>] + \exp(-\zeta(\lambda \\ &\qquad + 2\mu))[< 1^*|3^*>< 3^*|1^*>] = 1 \end{aligned}$$

$$P_{12}(\zeta) = < 1|A_\zeta|2>$$

$$= [< 1^*|1^* >< 1^*|2^* >] + \exp(-\zeta\mu)[< 1^*|2^* >< 2^*|2^* >] + \exp(-\zeta(\lambda + 2\mu))[< 1^*|3^* >< 3^*|2^* >] = 0$$

$$P_{13}(\zeta) =< 1\left|A_\zeta\right|3 >$$

$$= [< 1^*|1^* >< 1^*|3^* >] + \exp(-\zeta\mu)[< 1^*|2^* >< 2^*|3^* >] + \exp(-\zeta(\lambda + 2\mu))[< 1^*|3^* >< 3^*|3^* >] = 0$$

$$P_{21}(\zeta) =< 2\left|A_\zeta\right|1 >$$

$$= [< 2^*|1^* > +\lambda < 3^*|1^* >] < 1^*|1^* > +\exp(-\zeta\mu)[< 2^*|2^* > +\lambda < 3^*|2^* >] < 2^*|1^* > +\exp(-\zeta(\lambda + 2\mu))[< 2^*|3^* > +\lambda < 3^*|3^* >] < 3^*|1^* >= 0$$

$$P_{22}(\zeta) =< 2\left|A_\zeta\right|2 >$$

$$= \frac{1}{\Delta}[< 2^*|1^* > +\lambda < 3^*|1^* >][\mu < 1^*|2^* > +< 1^*|3^* >] + \frac{1}{\Delta}\exp(-\zeta\mu)[< 2^*|2^* > +\lambda < 3^*|2^* >][\mu < 2^*|2^* > +< 2^*|3^* >] + \frac{1}{\Delta}\exp(-\zeta(\lambda + 2\mu))[< 2^*|3^* > +\lambda < 3^*|3^* >][\mu < 3^*|2^* > +< 3^*|3^* >]$$

$$= \frac{1}{\Delta}\mu\exp(\zeta\mu) + \frac{1}{\Delta}\lambda\exp(\zeta(2\mu + \lambda)) = \frac{\exp(\zeta\mu)}{\Delta}(\mu + \lambda exp(\zeta\Delta))$$

$$P_{23}(\zeta) =< 2\left|A_\zeta\right|3 >$$

$$= \frac{1}{\Delta}[< 2^*|1^* > +\lambda < 3^*|1^* >][\lambda < 1^*|2^* > -< 1^*|3^* >] + \frac{1}{\Delta}\exp(-\zeta\mu)[< 2^*|2^* > +\lambda < 3^*|2^* >][\lambda < 2^*|2^* > -< 2^*|3^* >] + \frac{1}{\Delta}\exp(-\zeta(\lambda + 2\mu))[< 2^*|3^* > +\lambda < 3^*|3^* >][\lambda < 3^*|2^* > -< 3^*|3^* >]$$

$$= \frac{1}{\Delta}\lambda\exp(\zeta\mu) - \frac{1}{\Delta}\lambda\exp(\zeta(2\mu + \lambda)) = \frac{\lambda\exp(\zeta\mu)}{\Delta}(1 - exp(\zeta\Delta))$$

$$P_{31}(\zeta) =< 3\left|A_\zeta\right|1 >$$

$$= [< 3^*|1^* > -\mu < 3^*|1^* >] < 1^*|1^* > +\exp(-\zeta\mu)[< 2^*|2^* > -\mu < 3^*|2^* >] < 2^*|1^* > +\exp(-\zeta(\lambda + 2\mu))[< 2^*|3^* > -\mu < 3^*|3^* >] < 3^*|1^* >= 0$$

$$P_{32}(\zeta) =< 3\left|A_\zeta\right|2 >$$

$$= \frac{1}{\Delta}[< 2^*|1^* > -\mu < 3^*|1^* >][\mu < 1^*|2^* > +< 1^*|3^* >] + \frac{1}{\Delta}\exp(-\zeta\mu)[< 2^*|2^* > -\mu < 3^*|2^* >][\mu < 2^*|2^* > +< 2^*|3^* >] + \frac{1}{\Delta}\exp(-\zeta(\lambda + 2\mu))[< 2^*|3^* > -\mu < 3^*|3^* >][\mu < 3^*|2^* > +< 3^*|3^* >]$$

$$= \frac{1}{\Delta}\mu \exp(\zeta\mu) - \frac{1}{\Delta}\mu \exp(\zeta(2\mu+\lambda)) = \frac{\mu \exp(\zeta\mu)}{\Delta}(1 - exp(\zeta\Delta))$$

$P_{33}(\zeta) =< 3\left|A_\zeta\right|3 >$

$$= \frac{1}{\Delta}[< 2^*|1^* > -\mu < 3^*|1^* >][\lambda < 1^*|2^* > -< 1^*|3^* >]$$

$$+ \frac{1}{\Delta}\exp(-\zeta\mu)[< 2^*|2^* > -\mu < 3^*|2^* >][\lambda < 2^*|2^* > -< 2^*|3^* >]$$

$$+ \frac{1}{\Delta}\exp(-\zeta(\lambda+2\mu))[< 2^*|3^* > -\mu < 3^*|3^* >][\lambda < 3^*|2^* > -< 3^*|3^* >]$$

$$= \frac{1}{\Delta}\lambda \exp(\zeta\mu) + \frac{1}{\Delta}\mu \exp(\zeta(2\mu+\lambda)) = \frac{\exp(\zeta\mu)}{\Delta}(\lambda + \mu exp(\zeta\Delta)) \qquad (37)$$

Example 3.2: Let $H_1 = \begin{pmatrix} \lambda_1 & -\lambda_1 \\ -\mu_1 & \mu_1 \end{pmatrix} \in R^2$ and $H_2 = \begin{pmatrix} \lambda_2 & -\lambda_2 \\ -\mu_2 & \mu_2 \end{pmatrix} \in R^2$be two Hamiltonian matrices and $A_\zeta = \exp(-\zeta H) = \exp(-\zeta H_1) \otimes - \exp(-\zeta H_2) = \exp(-\zeta(H_1 \otimes I_2 + I_2 \otimes H_2))$ be a generated semi-group which consists of a tensor product. Then the elements of the non-ruin probability matrix are computed by algebraic calculation as seen in the solution.

Solution: In this example, similar calculations will be made. For brevity, all calculations will not be given. Now, I can start by using the notation $H = H_1 \otimes I_2 + I_2 \otimes H_2$. The matrix H is found as

$$H = \begin{bmatrix} \lambda_1+\lambda_2 & -\lambda_2 & -\lambda_1 & 0 \\ -\mu_2 & \lambda_1+\mu_2 & 0 & -\lambda_1 \\ -\mu_1 & 0 & \mu_1+\lambda_2 & -\lambda_2 \\ 0 & -\mu_1 & -\mu_2 & \mu_1+\mu_2 \end{bmatrix}$$

Let $|1^* >= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $|2^* >= \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ -\mu_1 \\ -\mu_1 \end{pmatrix}$, $|3^* >= \begin{pmatrix} \lambda_2 \\ -\mu_2 \\ \lambda_2 \\ -\mu_2 \end{pmatrix}$ and $|4^* >= \begin{pmatrix} \lambda_1\lambda_2 \\ -\lambda_1\mu_2 \\ -\lambda_2\mu_1 \\ \mu_1\mu_2 \end{pmatrix}$ be the eigenvectors of matrix H for the corresponding eigenvalues $K_1 = 0$, $K_2 = \lambda_1 + \mu_1$, $K_3 = \lambda_2 + \mu_2$ and $K_4 = \lambda_1 + \mu_1 + \lambda_2 + \mu_2$, respectively. For simplicity, $\Delta_1 = \lambda_1 + \mu_1$ and $\Delta_2 = \lambda_2 + \mu_2$ will be used in further calculations. Ket vectors $|1 >$, $|2 >$, $|3 >$ and $|4 >$ are calculated in the new coordinates as follows:

$$\left|1 >= (1 \quad 0 \quad 0 \quad 0)^T = \frac{\mu_1\mu_2}{\Delta_1\Delta_2}\right| 1^* > + \frac{\mu_2}{\Delta_1\Delta_2}\left|2^* > + \frac{\mu_1}{\Delta_1\Delta_2}\right| 3^* > + \frac{1}{\Delta_1\Delta_2}|4^* >,$$

$$\left|2 >= (0 \quad 1 \quad 0 \quad 0)^T = \frac{\mu_1\lambda_2}{\Delta_1\Delta_2}\right| 1^* > + \frac{\lambda_2}{\Delta_1\Delta_2}\left|2^* > - \frac{\mu_1}{\Delta_1\Delta_2}\right| 3^* > - \frac{1}{\Delta_1\Delta_2}|4^* >,$$

$$\left|3 >= (0 \quad 0 \quad 1 \quad 0)^T = \frac{\lambda_1\mu_2}{\Delta_1\Delta_2}\right| 1^* > - \frac{\mu_2}{\Delta_1\Delta_2}\left|2^* > + \frac{\lambda_1}{\Delta_1\Delta_2}\right| 3^* > - \frac{1}{\Delta_1\Delta_2}|4^* >,$$

$$\left|4 >= (0 \quad 0 \quad 0 \quad 1)^T = \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}\right| 1^* > - \frac{\lambda_2}{\Delta_1\Delta_2}\left|2^* > - \frac{\lambda_1}{\Delta_1\Delta_2}\right| 3^* > + \frac{1}{\Delta_1\Delta_2}|4^* >. \qquad (38)$$

To change the coordinates of vectors, I take matrix Σ as

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_1 & -\mu_1 & -\mu_1 \\ \lambda_2 & -\mu_2 & \lambda_2 & -\mu_2 \\ -\lambda_1\lambda_2 & \lambda_1\mu_2 & \lambda_2\mu_1 & -\mu_1\mu_2 \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \lambda_2 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 & -\mu_2 & \lambda_1\mu_2 \\ 1 & -\mu_1 & \lambda_2 & \lambda_2\mu_1 \\ 1 & -\mu_1 & -\mu_2 & -\mu_1\mu_2 \end{bmatrix} \tag{39}$$

Now, the following can be reached by applying the matrix to my ket vectors $|1>$, $|2>$, $|3>$ and $|4>$

$\Sigma 1 = |1^*> +\lambda_1|2^*> +\lambda_2|3^*> +\lambda_1\lambda_2|4^*>$,

$\Sigma 2 = |1^*> +\lambda_1|2^*> -\mu_2|3^*> -\lambda_1\mu_2|4^*>$,

$\Sigma 3 = |1^*> -\mu_1|2^*> +\lambda_2|3^*> -\lambda_2\mu_1|4^*>$,

$\Sigma 3 = |1^*> -\mu_1|2^*> -\mu_2|3^*> +\mu_1\mu_2|4^*>$.

Then, the operator in the new coordinates is defined by

$$< x|\exp(-\zeta H)|x' >=< x|\exp(-\zeta H)|\Sigma x' >_* \tag{40}$$

which equals to

$$< \Sigma x|\exp(-\zeta H)|1^* >_* < 1^*|x' >_* + < \Sigma x| \exp(-\zeta H) |2^* >_* < 2^*|x' >_*$$
$$+ < \Sigma x| \exp(-\zeta H) |3^* >_* < 3^*|x' >_* + < \Sigma x|\exp(-\zeta H)|4^* >_* < 4^*|x' >_*. \tag{41}$$

By applying eigenvectors to the matrix H, I get

$$H|1^* >= 0|1^* > , |2^* >= \Delta_1|2^* > , H|3^* >= \Delta_2|3^* > , H|4^* >= (\Delta_1+\Delta_2)|4^* >. \tag{42}$$

The operator is now derived as

$$< x|\exp(-\zeta H)|x' >=< \Sigma x|1^* >_* < 1^*|x' >_* + \exp(-\zeta\Delta_1)$$
$$< \Sigma x|2^* >_* < 2^*|x' >_* + \exp(-\zeta\Delta_2) < \Sigma x|3^* >_* < 3^*|x' >_*$$
$$+\exp(-\zeta(\Delta_1 + \Delta_2)) < \Sigma x|4^* >_* < 4^*|x' >_*. \tag{43}$$

Hence, the probabilities are calculated as follows:

$$P_{11}(\zeta) = \frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} + \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

$$P_{12}(\zeta) = \frac{\mu_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

$$P_{13}(\zeta) = \frac{\mu_2\lambda_1}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

$$P_{14}(\zeta) = \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

$$P_{21}(\zeta) = \frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_2)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} - \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}$$

$$P_{22}(\zeta) = \frac{\mu_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\zeta\Delta_2)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}$$

$$P_{23}(\zeta) = \frac{\mu_2\lambda_1}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} - \exp(-\zeta\Delta_2)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\zeta((\Delta_1 + \Delta_2))\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}$$

$$P_{24}(\zeta)=\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_2)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta((\Delta_1+\Delta_2))\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}$$
$$P_{31}(\zeta)=\frac{\mu_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}-\exp(-\zeta((\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$
$$P_{32}(\zeta)=\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}+\exp(-\zeta((\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$
$$P_{33}(\zeta)=\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}+\exp(-\zeta((\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$
$$P_{34}(\zeta)=\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}-\exp(-\zeta((\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$
$$P_{41}(\zeta)=\frac{\mu_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_2)\frac{\mu_1\mu_2}{\Delta_1\Delta_2}+\exp(-\zeta((\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$
$$P_{42}(\zeta)=\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_2)\frac{\mu_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta((\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$
$$P_{43}(\zeta)=\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta\Delta_2)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}-\exp(-\zeta((\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$
$$P_{44}(\zeta)=\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}+\exp(-\zeta\Delta_2)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2}+\exp(-\zeta((\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2} \tag{44}$$

The sum of rows is 1.

Example 3.3: Let $H_1=\begin{pmatrix}0&0&0\\0&-(\lambda+\mu)&\lambda\\0&\mu&-2\mu\end{pmatrix}\in R^2$ and $H_2=\begin{pmatrix}0&0&0\\0&-(\lambda+\mu)&\lambda\\0&\mu&-2\mu\end{pmatrix}\in R^2$ be two Hamiltonian matrices and $A_\zeta=\exp(-\zeta H)=\exp(-\zeta H_1)\otimes-\exp(-\zeta H_2)=\exp(-\zeta(H_1\otimes I_3+I_3\otimes H_2))$ be a generated semi-group, which consists of a tensor product. Then the elements of the non-ruin probability matrix are computed by algebraic calculation as seen in the solution.

Solution: In this example, similar calculations will be made. For brevity, all calculations will not be given. Now, I can start by using the notation $H=H_1\otimes I_3+I_3\otimes H_2$, the matrix H is found as

$$\begin{pmatrix}0&0&0&0&0&0&0&0&0\\0&-(\lambda+\mu)&\lambda&0&0&0&0&0&0\\0&\mu&-2\mu&0&0&0&0&0&0\\0&0&0&-(\lambda+\mu)&0&0&\lambda&0&0\\0&0&0&0&-2(\lambda+\mu)&\lambda&0&\lambda&0\\0&0&0&0&\mu&-(\lambda+3\mu)&0&0&\lambda\\0&0&0&\mu&0&0&-2\mu&0&0\\0&0&0&0&\mu&0&0&-(\lambda+3\mu)&\lambda\\0&0&0&\mu&0&\mu&0&\mu&-4\mu\end{pmatrix}$$

Let

$|1^*>=(1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1)^T$,

$|2^*>=(0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$

$|3^*>=(0 \quad \lambda \quad -\mu \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$

$|4^*>=(0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)^T,$

$|5^*>=(0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1)^T,$

$|6^*>=(0 \quad 0 \quad 0 \quad 0 \quad \lambda \quad -\mu \quad 0 \quad \lambda \quad -\mu)^T,$

$|7^*>=(0 \quad 0 \quad 0 \quad \lambda \quad 0 \quad 0 \quad -\mu \quad 0 \quad 0)^T,$

$|8^*>=(0 \quad 0 \quad 0 \quad 0 \quad \lambda \quad \lambda \quad 0 \quad -\mu \quad -\mu)^T,$

$|9^*>=(0 \quad \lambda \quad -\mu \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T$ (45)

be eigenvectors of matrix H for corresponding eigenvalues

$K_1 = 0,\ K_2 = K_4 = -\mu,\ \ K_3 = K_7 = -(\lambda + 2\mu),\ K_5 = -2\mu,$

$K_6 = K_8 = -(\lambda + 3\mu),\ K_9 = 2(\lambda + \mu),$ (46)

respectively. Ket vectors $|1>, |2>, |3>, |4>, |5>, |6>, |7>, |8>$ and $|9>$ are calculated in the new coordinates as follows

$|1>=|1^*>\ ,\ |2>=\frac{\mu}{\Delta}|2^*>+\frac{1}{\Delta}|3^*>\ ,\ \ |3>=\frac{\lambda}{\Delta}|2^*>-\frac{1}{\Delta}|3^*>\ ,$

$|4>=\frac{\mu}{\Delta}|4^*>+\frac{1}{\Delta}|7^*>\ ,\ |7>=\frac{\lambda}{\Delta}|4^*>-\frac{1}{\Delta}|7^*>$

$|5>=\frac{\mu^2}{\Delta^2}|5^*>+\frac{\mu}{\Delta^2}|6^*>+\frac{\mu}{\Delta^2}|8^*>+\frac{1}{\Delta^2}|9^*>\ ,$

$|6>=\frac{\lambda\mu}{\Delta^2}|5^*>-\frac{\mu}{\Delta^2}|6^*>+\frac{\lambda}{\Delta^2}|8^*>-\frac{1}{\Delta^2}|9^*>\ ,$

$|8>=\frac{\lambda\mu}{\Delta^2}|5^*>+\frac{\lambda}{\Delta^2}|6^*>-\frac{\mu}{\Delta^2}|8^*>-\frac{1}{\Delta^2}|9^*>\ ,$

$|9>=\frac{\lambda^2}{\Delta^2}|5^*>-\frac{\lambda}{\Delta^2}|6^*>-\frac{\lambda}{\Delta^2}|8^*>+\frac{1}{\Delta^2}|9^*>.$ (47)

To change the coordinates of vectors, I take matrix Σ as previous. Now, the following can be reached by applying the matrix to my ket vectors $|1>, |2>, |3>, |4>, |5>, |6>, |7>, |8>$ and $|9>$,

$\Sigma 1 = |1^*>\ ,\ \Sigma 2 = |2^*>+\lambda|3^*>\ ,\ \Sigma 3 = |2^*>-\mu|3^*>\ ,$

$\Sigma 4 = |4^*>+\lambda|3^*>\ ,\ \ \Sigma 7 = |4^*>-\mu|7^*>\ ,$

$\Sigma 5 = |5^*>+\lambda|6^*>+\lambda|8^*>+\lambda^2|9^*>\ ,$

$\Sigma 6 = |5^*>-\mu|6^*>+\lambda|8^*>-\lambda\mu|9^*>\ ,$

$\Sigma 8 = |5^*>+\lambda|6^*>-\mu|8^*>-\lambda\mu|9^*>\ ,$

$\Sigma 9 = |5^*>-\mu|6^*>-\mu|8^*>+\mu^2|9^*>.$ (48)

By applying eigenvectors to the matrix H, I get

$H|1^*>=0|1^*>, H|2^*>=\mu|2^*>, H|3^*>=-(\lambda+2\mu)|3^*>,\ H|4^*>=\mu|4^*>\ ,$

$H|5^*>=-2\mu|5^*>, H|6^*>=-(\lambda+3\mu)|6^*>, H|7^*>=-(\lambda+2\mu)|7^*>\ ,$

$$H|8^*>=-(\lambda+3\mu)|8^*>,\ H|9^*>=-2(\lambda+2\mu)|9^*>. \quad (49)$$

The operator is now derived as

$$<x|\exp(-\zeta H)|x'>=<\Sigma x|1^*>_*<1^*|x'>_*+\exp(-\zeta\mu)<\Sigma x|2^*>_*<2^*|x'>_*+$$
$$\exp\big(-\zeta(\lambda+2\mu)\big)<\Sigma x|3^*>_*<3^*|x'>_*+\exp(-\zeta\mu)<\Sigma x|4^*>_*<4^*|x'>_*+$$
$$\exp(-\zeta(2\mu))<\Sigma x|5^*>_*<5^*|x'>_*+\exp(-\zeta(\lambda+3\mu))<\Sigma x|6^*>_*<6^*|x'>_*+$$
$$\exp(-\zeta(\lambda+2\mu))<\Sigma x|7^*>_*<7^*|x'>_*+\exp(-\zeta(\lambda+3\mu))<\Sigma x|8^*>_*<8^*|x'>_*+$$
$$\exp(-\zeta(2\lambda+4\mu))<\Sigma x|9^*>_*<9^*|x'>_*. \quad (50)$$

Hence, the probabilities are calculated as follows:

$P_{11}(\zeta)=1,\quad P_{12}(\zeta)=0,\quad P_{13}(\zeta)=0,\quad P_{14}(\zeta)=0,\ P_{15}(\zeta)=0,\quad P_{16}(\zeta)=0,$
$P_{17}(\zeta)=0,\quad P_{18}(\zeta)=0,\quad P_{19}(\zeta)=0,$

$P_{21}(\zeta)=0,\quad P_{24}(\zeta)=0,\quad P_{25}(\zeta)=0,\quad P_{26}(\zeta)=0,\ P_{27}(\zeta)=0,\quad P_{28}(\zeta)=0,$
$P_{29}(\zeta)=0,$

$$P_{22}(\zeta)=\exp\left(-\tfrac{1}{2}\zeta\mu\right)\tfrac{\mu}{\Delta}+\exp\left(-\tfrac{1}{2}\zeta(\lambda+2\mu)\right)\tfrac{\lambda}{\Delta},$$

$$P_{23}(\zeta)=\exp\left(-\tfrac{1}{2}\zeta\mu\right)\tfrac{\lambda}{\Delta}-\exp\left(-\tfrac{1}{2}\zeta(\lambda+2\mu)\right)\tfrac{\lambda}{\Delta},$$

$P_{31}(\zeta)=0,\quad P_{34}(\zeta)=0,\quad P_{35}(\zeta)=0,\quad P_{36}(\zeta)=0,\ P_{37}(\zeta)=0,\quad P_{38}(\zeta)=0,$
$P_{39}(\zeta)=0,$

$$P_{32}(\zeta)=\exp\left(-\tfrac{1}{2}\zeta\mu\right)\tfrac{\mu}{\Delta}-\exp\left(-\tfrac{1}{2}\zeta(\lambda+2\mu)\right)\tfrac{\mu}{\Delta},$$

$$P_{33}(\zeta)=\exp\left(-\tfrac{1}{2}\zeta\mu\right)\tfrac{\lambda}{\Delta}+\exp\left(-\tfrac{1}{2}\zeta(\lambda+2\mu)\right)\tfrac{\mu}{\Delta},$$

$P_{41}(\zeta)=0,\quad P_{42}(\zeta)=0,\quad P_{43}(\zeta)=0,\quad P_{45}(\zeta)=0,\ P_{46}(\zeta)=0,\quad P_{48}(\zeta)=0,$
$P_{49}(\zeta)=0,$

$$P_{44}(\zeta)=\exp\left(-\tfrac{1}{2}\zeta\mu\right)\tfrac{\mu}{\Delta}+\exp\left(-\tfrac{1}{2}\zeta(\lambda+2\mu)\right)\tfrac{\lambda}{\Delta},$$

$$P_{47}(\zeta)=\exp\left(-\tfrac{1}{2}\zeta\mu\right)\tfrac{\lambda}{\Delta}-\exp\left(-\tfrac{1}{2}\zeta(\lambda+2\mu)\right)\tfrac{\lambda}{\Delta},$$

$P_{51}(\zeta)=0,\quad P_{52}(\zeta)=0,\quad P_{53}(\zeta)=0,\quad P_{54}(\zeta)=0,\ P_{57}(\zeta)=0,$

$$P_{55}(\zeta)=\exp(-\zeta(2\mu))\tfrac{\mu^2}{\Delta^2}+\exp\left(-\tfrac{1}{2}\zeta(\lambda+3\mu)\right)\tfrac{\lambda\mu}{\Delta^2}+\exp\left(-\tfrac{1}{2}\zeta(\lambda+3\mu)\right)\tfrac{\lambda\mu}{\Delta^2}+$$
$$\exp\left(-\tfrac{1}{2}\zeta(2\lambda+4\mu)\right)\tfrac{\lambda^2}{\Delta^2},$$

$$P_{56}(\zeta)=\exp(-\zeta(2\mu))\tfrac{\lambda\mu}{\Delta^2}-\exp\left(-\tfrac{1}{2}\zeta(\lambda+3\mu)\right)\tfrac{\lambda\mu}{\Delta^2}+\exp\left(-\tfrac{1}{2}\zeta(\lambda+3\mu)\right)\tfrac{\lambda^2}{\Delta^2}-$$
$$\exp\left(-\tfrac{1}{2}\zeta(2\lambda+4\mu)\right)\tfrac{\lambda^2}{\Delta^2},$$

$P_{58}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} -$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda^2}{\Delta^2},$

$P_{59}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} +$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda^2}{\Delta^2},$

$P_{61}(\zeta) = 0, \quad P_{62}(\zeta) = 0, \quad P_{63}(\zeta) = 0, \quad P_{64}(\zeta) = 0, \; P_{67}(\zeta) = 0,$

$P_{65}(\zeta) = \exp(-\zeta(2\mu))\frac{\mu^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} -$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{66}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} +$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{68}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} +$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{69}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} +$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{71}(\zeta) = 0, \quad P_{72}(\zeta) = 0, \quad P_{73}(\zeta) = 0, \quad P_{75}(\zeta) = 0, \; P_{76}(\zeta) = 0, \quad P_{78}(\zeta) = 0,$

$P_{79}(\zeta) = 0,$

$P_{74}(\zeta) = \exp\left(-\frac{1}{2}\zeta\mu\right)\frac{\mu}{\Delta} - \exp\left(-\frac{1}{2}\zeta(\lambda+2\mu)\right)\frac{\mu}{\Delta},$

$P_{77}(\zeta) = \exp\left(-\frac{1}{2}\zeta\mu\right)\frac{\lambda}{\Delta} + \exp\left(-\frac{1}{2}\zeta(\lambda+2\mu)\right)\frac{\mu}{\Delta},$

$P_{81}(\zeta) = 0, \quad P_{82}(\zeta) = 0, \quad P_{83}(\zeta) = 0, \quad P_{84}(\zeta) = 0, \; P_{87}(\zeta) = 0,$

$P_{85}(\zeta) = \exp(-\zeta(2\mu))\frac{\mu^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} +$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{86}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} -$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{88}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} +$

$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{89}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} -$
$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\lambda\mu}{\Delta^2},$

$P_{91}(\zeta) = 0, \quad P_{92}(\zeta) = 0, \quad P_{93}(\zeta) = 0, \quad P_{94}(\zeta) = 0, \; P_{97}(\zeta) = 0,$

$P_{95}(\zeta) = \exp(-\zeta(2\mu))\frac{\mu^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} +$
$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\mu^2}{\Delta^2},$

$P_{96}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} -$
$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\mu^2}{\Delta^2},$

$P_{98}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda\mu}{\Delta^2} - \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\mu^2}{\Delta^2} -$
$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\mu^2}{\Delta^2},$

$P_{99}(\zeta) = \exp(-\zeta(2\mu))\frac{\lambda^2}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} + \exp\left(-\frac{1}{2}\zeta(\lambda+3\mu)\right)\frac{\lambda\mu}{\Delta^2} +$
$\exp\left(-\frac{1}{2}\zeta(2\lambda+4\mu)\right)\frac{\mu^2}{\Delta^2}.$

4. CONCLUSIONS

In this study, I concentrate on Markovian structure and apply the discrete and continuous space quantum mechanics formalism with the appropriately chosen Hamiltonians to compute finite time non-ruin probability. Finally, several advanced examples are treated for chosen Hamiltonians in discrete and continuous space using traditional bases, as well as eigenvalues of Hermitian operators in two and three dimensions using the tensor product of operators and standard Dirac matrix formalism with bra-ket notations.

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REFERENCES

[1] S. Asmussen and H. Albrecher, *Ruin Probabilities (Advanced Statistical Science and Applied Probability)*, Second Edition (World Scientific, Singapore, **2010**) pp. 19-30.

[2] H. U. Gerber, *An Introduction to Mathematical Risk Theory*, First Edition (Huebner Foundation Monograph, R. D. Irwin Inc. Homeward Illinois, **1979**) pp. 16-34, 110-113.

[3] C. Lefèvre, S. Loisel, M. Tamturk, S. Utev, *A Quantum-Type Approach to Non-Life Insurance Risk Modelling*, Risks, 6(3) (**2018**) 99.

[4] P. A. M. Dirac, *Hamiltonian Methods and Quantum Mechanics*, Proceedings of the Royal Irish Academy, Section A: Mathematical and Physical Sciences, 63 (**1963**) 49-59.

[5] B. E. Baaquie, *Quantum Mechanics and Option Pricing.* In: Proceedings of the Second Quantum Interaction Symposium (QI-2008), (**2008**) 54-59.

[6] B. E. Baaquie, *The Theoretical Foundations of Quantum Mechanics*, First Edition (Springer New York, New York, **2013**).

[7] D. J. Griffiths, *Introduction to Quantum Mechanics*, Second Edition (Pearson Prentice Hall, New Jersey, **2004**) pp. 1-170.

[8] J. Dimock, *Quantum Mechanics and Quantum Field Theory: a Mathematical Primer*, First Edition (Cambridge University Press, New York, **2011**).

[9] R. P. Feynman, *Space-time Approach to Non-relativistic Quantum Mechanics*, Reviews of Modern Physics 20(2) (**1948**) 367.

[10] S. M. Ross, *Introduction to Probability Models*, Tenth Edition (Elsevier Academic Press, Oxford, **2010**).

[11] S. M. Ross, *Applied Probability Models with Optimization Applicat*ion, First Edition (Holden-Day, San Francisco, **1970**) pp. 10-25, 61-65.

[12] T. Mikosch, *Non-Life Insurance Mathematics*, Second Edition (Springer, Heidelberg, **2009**) pp. 7-18, 155-159.

[13] C. Lefèvre, and S. Loisel, *On Finite-time Ruin Probabilities for Classical Risk Models*, Scandinavian Actuarial Journal 2008(1) (**2008**) 41-60.

[14] J. R. Norris, *Markov Chains*, First Edition (Cambridge University Press, Cambridge, **1997**), pp. Preface-10, 73-81.

[15] S. Jain, *Markov Chain Model and Its Application*, Computers and Biomedical Research, 19(4) (**1986**) 374-378.

[16] Y. Suhov, and M. Kelbert, *Probability and Statistics by Example: Volume 2, Markov Chains: a Primer in Random Processes and Their Applications*, (Cambridge University Press, New York, **2008**) pp. 1-17, 185-230.

[17] J. J. Sakurai, *Modern Quantum Mechanics*, (Addison-Wesley Publishing Company, **1994**).

[18] K. L. Chang, *Mathematical Structures of Quantum Mechanics*, (World Scientific, Singapore, **2012**).

[19] M. S. Bespalov, *On the Properties of a New Tensor Product of Matrices*, Computational Mathematics and Mathematical Physics, 54(4) (**2014**) 561-574.

[20] A. Kaya, *Modern Mathematical Methods for Actuarial Sciences*, (University of Leicester PhD Thesis. https://hdl.handle.net/2381/39613, **2007**).